STUDENTS AND DECIMAL NOTATION: DO THEY SEE WHAT WE SEE?

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While a test based on comparing pairs of decimal numbers seems to address only a small portion of the curriculum, it is the pattern of errors that students make on this task that allows us to hypothesize what they may be thinking about decimal numbers. A student who holds an incorrect view of what decimals *are* will probably spend most of their schooling rote-learning meaningless algorithms for the manipulation and comparison of decimals. The important skill of estimation is disabled if any problem involving decimals creates a sense of disorientation. Rounding numbers and using scientific notation will also be meaningless activities.

The results of testing 2517 students in 1997 with the Decimal Comparison Test are briefly presented in Table 1; for a more detailed discussion see Steinle & Stacey (1998). The 30 question test was given to students at 6 primary and 7 secondary schools across Melbourne. The numbers of students in each Grade/Year level are indicated, as well as the percentages of students who use comparison strategies based on the *lengths* of the decimals involved.

Table 1
Average percentage of students by grade/year level and by classification

Grade/ Year level	Number of students tested (Total 2517)	Longer-is-larger misconceptions	Shorter-is-larger misconceptions	Task Expert	Other
5	294	32%	15%	23%	30%
6	319	17%	12%	52%	19%
7	814	13%	14%	54%	19%
8	457	9%	16%	49%	26%
9	350	6%	10%	58%	26%
10	283	5%	11%	58%	26%

Students in the 'Longer-is-larger' misconception group are most probably using *Whole number thinking* where 4.63 is larger than 4.8 as 63 > 8. We have also found evidence for the more advanced *Right hand overflow thinking* who may choose 4.63 > 4.8 as 63 tenths > 8 tenths, but who make the correct choice on 4.7 and 4.08 as 7 tenths > 8 hundredths. The decreasing trend (32% of Grade 5 to 5% of Year 10) suggests that these ways of thinking are not common in adults.

Students in the 'Shorter-is-larger' misconception group are likely to be using one of the following three ways of thinking: *Denominator focussed thinking* (incorrectly generalising the fact that 1 tenth is bigger than 1 hundredth to 'any number of tenths is bigger than any number of hundredths'), *Reciprocal thinking* (0.3 is bigger then 0.4 as 1/3 is bigger than 1/4), or *Negative thinking* (0.3 is bigger then 0.4 as -3 is bigger than -4). Evidence for the last two ways of thinking was found in interviews of tertiary students. It is rather disturbing that the size of this group is between 10% and 16% at all year levels tested. Although the results presented are from a cross-sectional study it appears that these misconceptions are held onto tenaciously. We expect the results from the longitudinal study will confirm this.

The group labelled ‘Task Expert’ refers to students who make very few mistakes on the 30 item test; but no claim is made on their understanding of all decimal concepts. Within this group (as evidenced in interviews) are students who are blindly following rules and whose understanding of decimals may in fact be quite limited. It is of concern to us that while only 58% of Year 10 students can complete the test in this manner, very little gains have been made since the end of primary school (52% of Grade 6 are Task experts). So approximately 40% of students in Years 7 to 10 do not possess an interpretation of decimals which allows them to correctly (and consistently) order a pair of decimal numbers. If they were not able to do this by the end of their primary schooling, then they are unlikely to develop this skill in the next four years at school.

The ‘Other’ group consists mostly of students who were unclassified by the test; their incorrect responses were scattered throughout the test instead of being confined to particular types of comparisons. Also included in this group are students who are only able to cope with comparisons where the decimals have unequal digits in the first 2 places. Such students (about 4% of the sample) may view decimals as two numbers separated by a dot; the first possibly representing dollars and the second cents (again an interview with a tertiary student provided this evidence) and have been labelled *Money thinkers.* The size of this group (20% to 30% of each year level) illustrates that about one quarter of students remain uncertain about the nature of decimal numbers.

We believe that part of the reason that students have so much difficulty understanding the notation of decimals is due to the sequence of teaching. The study of decimals usually starts with decimals of length 1 (tenths) followed by length 2 (hundredths) and then some examples of decimals of lengths 3 or more. This reminds them of their experiences with whole numbers which were learnt in a similar manner and *in order of size*. With decimals, however, longer decimals can fit between shorter decimals (eg 0.35 lies between 0.3 and 0.4). If a student tries to sequence the decimals of different lengths using the same order that worked in the domain of whole numbers, then the student will be diagnosed with a Longer-is-larger misconception, typically *Whole number thinking.* If however, they sequence the different groups of decimals in the reverse order (i.e. length 1 is bigger than length 2, etc.), they will be using *Reciprocal thinking* or other Shorter-is-larger misconceptions such as *Denominator focussed thinking* or *Negative thinking.*

Diagnose your own students

It is possible to diagnose how your students think about decimal numbers by giving the short decimal comparison test in Table 2. In order to use this test, simply mix the decimal pairs so that the correct decimal is not always on the left and remove the horizontal lines between the different groups of decimals. Note that as this is a diagnostic test, the total number of correct answers is a meaningless quantity.

Table 2
Decimal Comparison Test with samples of error patterns

	Decimal Comparisons	Sample 1 Whole number thinking	Sample 2 Denominator focussed thinking	Sample 3 Reciprocal or Negative thinking	Sample 4 Money thinking
1	4.8 / 4.63	I	C	C	C
	0.5 / 0.36	I	C	C	C
	0.8 / 0.76	I	C	C	C
	2.37 / 2.126	I	C	C	C
2	5.736 / 5.62	C	I	I	C
	0.73 / 0.6	C	I	I	C
	0.42 / 0.3	C	I	I	C
	2.832 / 2.59	C	I	I	C
3	4.358 / 4.35	C	I	I	any errors here
	1.382 / 1.38	C	I	I	
4	0.4 / 0.3	C	C	I	C
	1.85 / 1.84	C	C	I	C

C = correct, I = incorrect

Four samples have been provided for matching with your students' tests. A student using *Whole number thinking* will choose incorrectly on the first 4 items (Group 1), correctly on the next 4 items (Group 2) and correctly again on Groups 3 and 4 (see Sample 1). Students using *Denominator focussed thinking* or *Reciprocal / Negative thinking* will be correct on Group 1 and incorrect on Groups 2 and 3; their performance on Group 4 allows them to be separated (see Samples 2 and 3). Note that students who follow Sample 3 may be using either *Reciprocal thinking* or *Negative thinking* and they may then be separated by including additional test items which include 0 as one of the decimals, or by using a question which involved marking decimals on a number line. A student who only makes mistakes on Group 3 is likely to be using *Money thinking,* (see Sample 4) while a student making no or few errors is classed as *Task expert* .

What to do

In this section we will present various activities which could be used to assist students overcome their misunderstanding of the notation used for decimals. Firstly, an activity called Marking Homework is discussed which encourages students to try to diagnose how another child interprets decimals. After testing students to find the most prevalent misconception, teachers could choose which of the Marking Homework samples are relevant to their classes. While these may be most useful to students who find their own (wrong) way of thinking being openly discussed and found to be faulty, this activity may also help those who haven't been classified by the test. It is reassuring that Swan (1983a) found in his study that "there were no students who regressed when they were introduced to misconceptions that they did not possess" (p1).

Secondly, activities which should help all students with incorrect ideas are discussed. These include references to the *Endless Base 10 Chain,* (see Figure 1), which is one of the important properties of the Base 10 numeration system that seems to be overlooked by students with misconceptions. These and many more activities are available on the website "Teaching and Learning about Decimal Numbers" which can be accessed by contacting the authors.

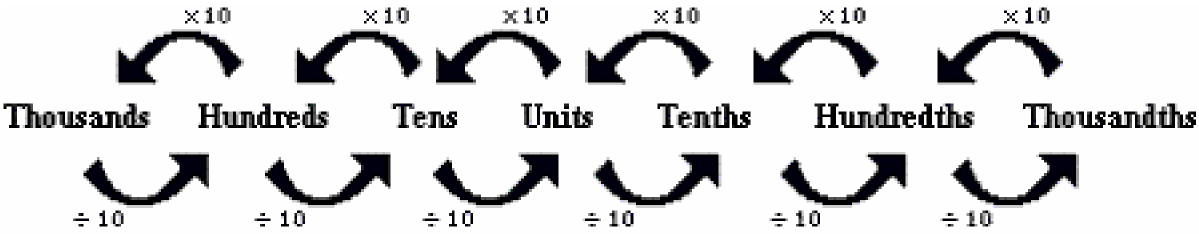

Figure 1. Endless Base 10 Chain

Marking Homework

Caitlin, a *Whole Number thinker*, is a fictitious child. She is a compilation of children discussed in research papers and found in many classrooms. 'Her' attempts at completing a homework sheet are given in Figure 2, with the instructions "Caitlin has made some mistakes with her homework. Can you find them?". Students may use calculators to check her answers and should provide corrections to *help Caitlin*. The questions at the end of the homework "Can you work out what she was thinking?" and "How would you explain decimal numbers to Caitlin so that she could understand?" are designed to stimulate group discussion on what decimals *are not* as well as what decimals *are*. Marking Homework samples for Ricardo (*Denominator Focussed thinking*), Courtney (*Reciprocal thinking*) and Maria (*Money thinking*) are given in Figures 3, 4 and 5 respectively.

(The following figures are available from the Marking Homework activity)

Figure 2. Marking Homework: Caitlin- Whole Number thinking

Figure 3. Marking Homework: Ricardo- Denominator Focussed thinking

Figure 4. Marking Homework: Courtney- Reciprocal thinking

Figure 5. Marking Homework: Maria- Money thinking

Alien Invaders

Cheeseman (1994) describes a calculator game based on that reported by Swan (1983b). A number composed of any of the digits and a decimal point is entered into a calculator by an opponent (eg 63724.518). The "aliens" (digits) need to be "shot down" (changed to zero) in the correct numerical order for the student to win, however, only aliens in the units column can be hit (by using the subtract button). In order to move the target digit into the units column, multiplication and division by 10 and it's powers are required. Hence the first steps in this game would be x100, -1, ÷ 1000, -2, etc. The game could be limited by stating the number of digits to enter or by not allowing the use of division (hence multiplication by decimals like 0.00001 is required). This game can be used to illustrate the relationship between the place value columns and the Endless Base 10 Chain property. Unfortunately, the display on some calculators may reinforce the idea that it is the decimal point which moves rather than the digits moving into new columns. It would therefore be an appropriate time to demonstrate the effect of multiplication and division by powers of ten on a number slide (see Figure 6). The digits of the number are written on a strip woven through a frame. As the strip is pulled sideways, the digits move to new columns. Appreciating the reason for the movement of the digits between the columns may reduce the reliance on, and misuse of, rules about moving a decimal point. This game may help students with *Whole number thinking* and *Money thinking* and possibly others.

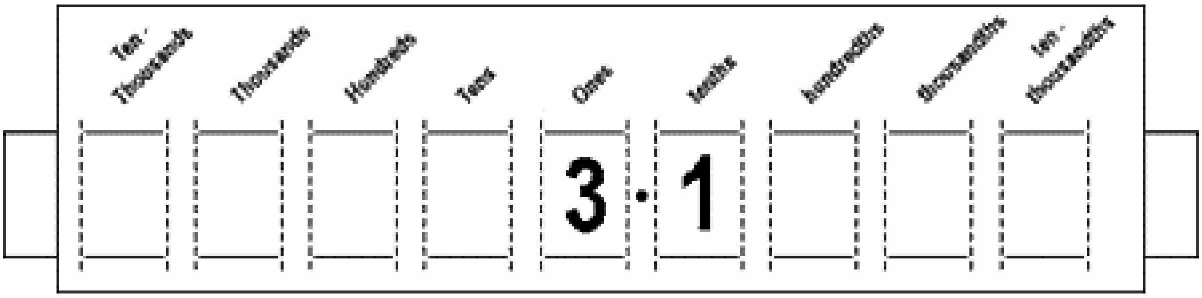

Figure 6. A number slide

Swan (1983b) describes a different version of this game; the spaceships are shot down (again in ascending order) but remain in their original columns. For the example 63724.518, the first steps are -0.01, -20, -3000, etc. Students could work in pairs and verbalise the numbers that they enter; ensure that the names of the columns are being used (5 tenths) rather than the digit by digit “spelling” (point 5). This version of Alien Invaders may assist students using *Reciprocal thinking,* as well as *Whole Number and Money thinking*.

In both versions of this game, a table (consisting of 2 columns) could be used to record the calculator display after the keys are pressed. This will allow students to reflect on their mistakes as well as consider the role of zeros in recording numbers. In the example given in the second game, the display after 5 steps is 60700.008, followed by 700.008, 0.008 and then 0.

Number Between Game

To start this game, the teacher could ask a student to choose any 2 numbers. These are marked on a rough numberline on the board. Note that the numberline could be drawn horizontally (as is done in most printed material to conserve space) or vertically which appeals to the natural link between the words “more” and “up”. (Students require both vertical and horizontal numberlines for co-ordinate geometry). A student is then chosen to give any number which lies between the starting points (not necessarily the midpoint). The teacher then decides which of the two subintervals to choose and the process is repeated. This game must eventually lead to decimals; for example a game starting with -112 and 560 may end up dealing with numbers between 135.47 and 135.48. This game is used to link several sets of numbers that tend to be presented separately, i.e. whole numbers, decimals and negative numbers. This game should assist all students to improve their understanding of decimal notation.

Linear Arithmetic Blocks (L.A.B.)

This is a concrete model which was described to us by Heather McCarthy (see Figure 7). It allows units, tenths, hundredths and thousandths to be modelled in a similar way to MAB but it has two advantages. Firstly, as it is a different model, students may more easily accept that the numbers represented are decimals. Secondly, it is a linear model rather than a volume model and when the LAB pieces are removed from the organiser (rods which can hold the washers and pieces of tubing in the correct columns) the individual pieces can be arranged in a line and hence effectively model the numberline. Use of this model should assist most students, as many find the concept of the numberline hard to grasp yet it is the model which students needs to carry with them into adult life (in the form of a measuring tape).

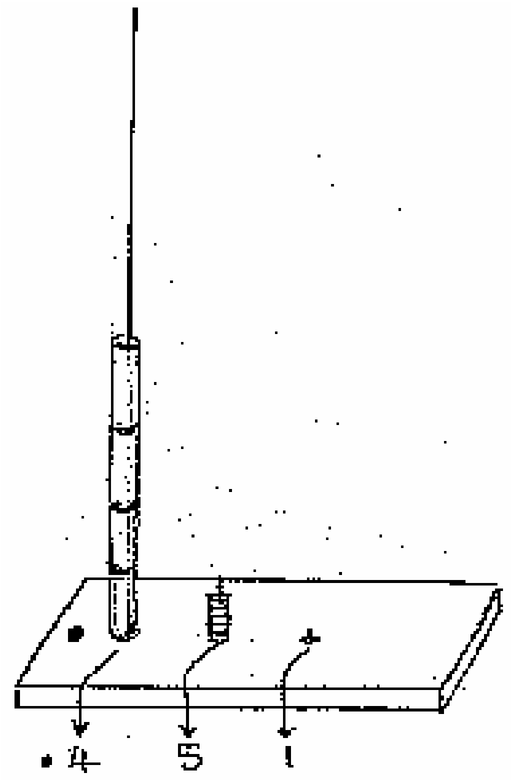

Figure 7. Linear Arithmetic Blocks

Conclusion

Dealing with "ragged" decimals can easily be overlooked in the standard curriculum, yet it is exactly this that provides a window into children's concepts of decimal notation. Fundamental misconceptions are not challenged if children only deal with decimals of equal length. The activities and tests described above should provoke many students to question and reorganise their current understandings.

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REFERENCES

Cheeseman, J. (1994). Making Sense of Decimals- How Can Calculators Help? In Cathy Beesey, Duncan Rasmussen. (Eds) *Mathematics Without Limits, MAV 94.* pp169-172. Mathematics Association of Victoria.

Stacey, K. & Steinle, V. (1998). Refining the Classification of Students' Interpretations of Decimal Notation. *Hiroshima Journal of Mathematics Education, 6*, 49-69.

Steinle, V. & Stacey, K. (1998). The incidence of misconceptions of decimal notation amongst students in Grades 5 to 10. In Clive Kanes, Merrilyn Goos, Elizabeth Warren. (Eds). *Teaching Mathematics in New Times, MERGA 21.* Volume 2, pp548-555. Mathematics Education Research Group of Australasia.

Swan, M. (1983a). Teaching Decimal Place Value: A Comparative Study of "Conflict" and "Positive only" Approaches. Nottingham: Shell Centre for Mathematics Education.

Swan, M. (1983b). The Meaning and Use of Decimals (Pilot version). Nottingham: Shell Centre for Mathematics Education.