

USING DYNAMIC DIAGRAMS TO MOTIVATE AND ASSIST PROBLEM SOLVING BY ALGEBRA

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Dynamic diagrams can provide stimulating teaching and learning opportunities that may lead to improved understandings of the concepts of variables and functions, afford a clearer understanding of the meaning of problem components and motivate more and better use of algebra as an aid to their solution. Presenting diagrams of this type in a book, out of necessity, makes them static rather than dynamic. Here we can only describe, rather than visually demonstrate, how these diagrams are engaging, dynamic and changeable.

Introduction

Diagrams have always played an important role in the creation and communication of mathematical ideas and thinking. They are useful to the mathematician in the initial stage of discovery and in the later stage of communication of the discovered ideas and the reasoning and proof that justifies their correctness (HREF1). Diagrams are also vital in the later teaching and learning of this mathematical knowledge. Here teachers use them to assist the development of student understanding of the mathematics and its application in solving problems.

Here too, students should learn to create and use diagrams to explore, discover and verify results previously unknown to them and, in so doing, gain a deeper understanding of key ideas, skills and procedures and their use in finding solutions to problems (HREF2, HREF3).

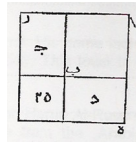
With the advent of computers and special purpose software the nature of the diagrams available for use by students and teachers of mathematics has progressed a long way from the “wetted sand, dusted surface or wax tablets” known and used by the Greeks (Netz, 1999); and even from the paper and pencil, chalk and chalkboard, pen and overhead transparency, marker and whiteboard diagrams most of us have known and used. Diagrams constructed with this old technology remain useful but they are static. With the new technology, diagrams can become dynamic and able to be manipulated. In this paper we present and describe examples of different types of diagrams that can be created with dynamic geometry programs and discuss how they might be used in the classroom. Several excellent dynamic geometry programs are available including *Cabri Geometry* (HREF4) and *DynaGeo* (HREF5). In this paper and presentation we have used the dynamic geometry program *The Geometer’s Sketchpad* (HREF6). Of course, dynamic geometry is a wonderful tool for teaching about geometry, but in this paper we will show uses in teaching algebra.

A Static Diagram

Using a dynamic geometry program it is possible to make diagrams similar to those made with the older technology, and which serve similar teaching purposes. These diagrams can have some degree of changeability but remain largely static. They can be used in conjunction with written or oral dialogue in the presentation of ideas or steps in a mathematical procedure. They can also be used to depict a problem and the key information known about the problem, but without providing the possibility of altering the size of geometric components to change the number values for their measurements. We will illustrate this type of diagram using a well known problem from the history of mathematics.

The treatise *Hisab al-jabr w'al-muqabala*, written around 820 A.D., is probably the most important work of the Persian mathematician Muhammad al-Khwarizmi, and it is from its title that we get the word algebra. In this treatise al-Khwarizmi poses the problem given in Figure 1 and uses written text to describe how the positive number solution could be found by the procedure we know as “completing the square”. He included the ‘figure’ shown in Figure 1 to help communicate and give meaning to what he had written.

Al-Khwarizimi used the 'square' in the upper right-hand corner of the figure to represent the one square. To this square he added "ten of its own roots" as two rectangles with side of five. This is shown in his figure as the rectangles on two sides of the square (the bottom and left-hand sides). The next paragraph paraphrases what al- Khwarizimi wrote.



Problem : One square, and ten roots of the same, amount to thirty-nine dirhems; that is to say, what must be the square which when increased by ten of its own roots, amounts to thirty-nine?

Figure 1. Original Figure Used by al-Khwarizimi and Problem

Each rectangle has a length of five and "breadth equal to a side of the square". The one square and two rectangles together make up the thirty-nine. The "great" square in Figure 1 is formed when the small square in the lower left-hand corner is added to the one square and two rectangles, and twenty-five is added to the thirty-nine. The great square now represents sixty-four, so its side is eight, and hence the side of the original small square must be eight minus five, which is three.

If we were to use this problem we would probably do and say something similar to al-Khwarizimi. The differences would be in the use of possibly more succinct written and/or oral language, and most certainly in the use of symbolic notation to set out the steps in the procedure. We could use a diagram like al-Khwarizimi's, but a better alternative would be a diagram that incorporates colour to visually engage students, and to highlight and to link important aspects of the setting out of the procedure with their associated elements in the diagram. Figure 2 shows the final view of a possible *Geometer's Sketchpad* sketch that does this.

A way to solve the quadratic equation $x^2 + 10x = 39$

Click these buttons in the order given.

[Show Diagram and Labels](#)

[Complete to Make a Big Square](#)

[Outline Big Square, Factor and Solve](#)

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25$$

$$(x + 5)(x + 5) = 64$$

$$(x + 5)^2 = 8^2$$

$$x + 5 = 8$$

$$x = 3$$

Figure 2. Sketch for Completing the Square

This sketch uses “hide/show buttons” to provide control to the teacher (or student) so that the solution steps and relevant parts of the developing diagram can be jointly sequenced to arise and be “seen” (understood) as the accompanying dialogue proceeds from the representation of the problem to its final solution. Something similar to this could be done with general presentation software like PowerPoint. But the advantage with most dynamic geometry software is the availability of text, mathematical symbols and geometrical figures in the one software package.

Al-Khwarizimi’s problem can still be used with year 9 or 10 students to have them learn about “completing the square”. In fact, Glaubitz (HREF7) used it with year 9 students in a study of a historico-mathematical teaching approach. He found that students taught with this approach showed greater improvement in skills than students taught without the historical focus.

A Changeable Diagram

A changeable diagram for a problem is a diagram that depicts and maintains the general meaning and essential features of the problem, but allows key components in the diagram to change. Diagrams of this type can help students understand what a problem is asking them to find and can provide them with a way to test the correctness of an answer. A changeable diagram can be particularly effective in helping students “see” how a change in one quantity brings about changes in others and what it means for a problem to be generalised. By viewing the varying images produced by a changeable diagram student’s interest in the problem can be aroused and maintained, thereby encouraging their engagement with the problem and possibly even motivating them to want to solve it.

An example of a changeable diagram is shown in Figure 3. It was created for the problem given below. This problem is suitable for year 10 or year 11 students, and is best done with the aid of CAS, particularly for year 10 students.

- (a) Points $O(0,0)$ and $P(m,n)$ are two fixed points with $m > 0$ and $n > 0$. There is a family of parabolas that pass through O and P with their maximum to the left of P . Each parabola in this family will intersect the x -axis in a point $Q(k,0)$ for some k . Find a general rule for the family of parabolas. What restrictions must be placed on the number values for k ?
- (b) Find the general rule for the family of parabolas that are the reflection in the line $x = m$ of the parabolas in part (a).
- (c) For $m = 4$ and $n = 8$ choose any three of the possible values for k . With the aid of a graphing or a CAS calculator draw on the same set of axes the graph of the parabola and its reflection for each of the values of k .
- (d) The maximum points for all the parabolas in part (a) lie on the graph of a function. Find the rule for this function.

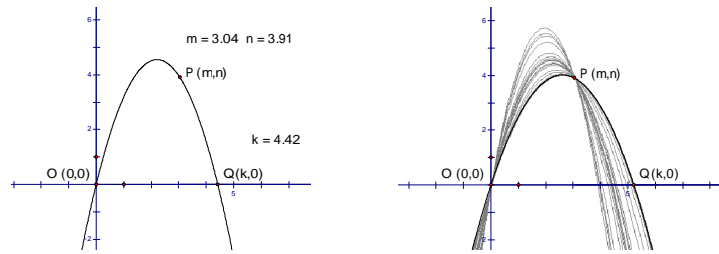


Figure 3. Parabola Sketch and Some Parabolas in the Family

To make the Parabola Sketch in Figure 3 the point P was constructed as a ‘free point’ in the first quadrant. When P is dragged to a location in this quadrant the values of m and n are set. To ensure that the maximum of the parabola lies to the left of the point P we must have $m < k < 2m$. Therefore, the point Q was constructed as a free point on the segment joining the points $(m,0)$ and $(2m,0)$.

To produce the changeable parabola in the Parabola Sketch the rule for the general parabola is needed. This rule can be found by simultaneously solving the three equations that result when the coordinates of the points O , P and Q are substituted into $f(x) = ax^2 + bx + c$. The rule for the general parabola is:

$$f(x) = -\frac{n}{m(k-m)}x^2 + \frac{kn}{m(k-m)}x \quad \text{where } m > 0, n > 0, m < k < 2m$$

Next the values for m , n and k were measured on the Parabola Sketch and the function plotting feature of *Geometer's Sketchpad* was used with these measurements and the general quadratic rule to plot its graph. This completed Sketch is what should be shown to and used with (by) students. Teacher algebra needs to be done first in order to construct the changeable diagram. The diagram is then used by the teacher with the students to motivate and encourage the use of algebra by the students.

On the Parabola Sketch the point Q can be dragged to see the effect of changing the values of k . This would assist students to see that every different value for k produces a different parabola through the two points O and P , and also to understand why the values of k need to be restricted to ensure that the parabola's maximum lies to the left of the point P .

Seeing a recorded display of some of the possible parabolas in the family of parabolas can also help students gain an appreciation of the general nature of the problem. Clicking on the parabola showing on the Sketch, selecting its function plot to be traced and then dragging point P was done to produce some of the parabolas in the family in Figure 3.

When students think they have found the correct rule for the reflected parabola that is asked for in part (b), the Parabola Sketch can be used to verify whether or not their rule is correct. This is done by plotting their rule for the displayed and measured values of m , n and k to see if the required parabola is produced. See Figure 4. Also point P or point Q can be dragged after the correct rule has been plotted to confirm that the reflection relationship between the two parabolas is maintained when m , n and k change.

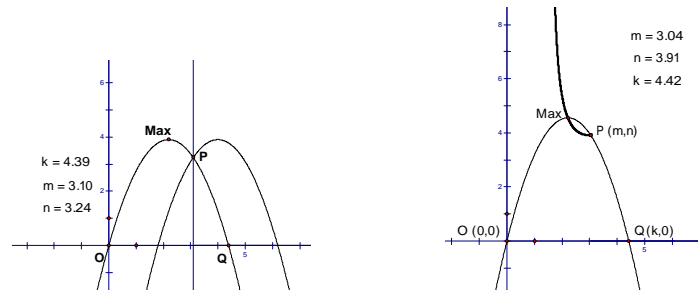


Figure 4. Parabola with Reflection and Parabola with Locus of Maximum Point

Finally, the Parabola Sketch can be used to demonstrate that the maximum points for the parabolas in part (a) lie on the graph of a function. To do this $k/2$ and $f(k/2)$ are computed using *Geometer's Sketchpad's* calculator. Next $(k/2, f(k/2))$ is plotted and its locus produced by *Sketchpad* as the point Q changes through all its possible values. Figure 4 shows the locus. The challenge for students is to find the algebraic rule for the function on which this locus lies. The function rule can be found by substituting $k = 2x$ in $y = f(k/2)$. When students find what they believe is the correct function rule it can be plotted on the Parabola Sketch to check to see if the locus is part of the graph of the function. Note that the function for the locus is not a parabola, as the rule for the function is:

$$y = \frac{nx^2}{m(2x - m)}$$

Dynamic Diagrams to Simulate Problems

Sketchpad can also be used to create diagrams that depict a problem and also allow its components to be measured and dynamically altered to produce changeable and relationally linked data that can be explored by means of tables and graphs generated from the data. This type of diagram is essentially a scaled simulation of a problem situation. They can be the stimulus for the generation of algebraic representations of the functions for the related measurements and their graphs, and the use of the function rules in solving the problems posed.

Two problems for year 9/10 students that have been used in the RITEMATHS Project (HREF7) are given below. Their *Geometer's Sketchpad* sketches are examples of the simulation type of dynamic diagram. See Figure 5 and Figure 6 for the final view of each sketch. The two sketches and an accompanying student worksheet for the Danish Flag problem are available from HREF8.

Danish Flag Problem: The Danish flag has a white cross over a red background. Both stripes of the white cross always have the same width. For the celebration of Princess Alexandra's birthday, the Royal Family ordered a special flag with these requirements: the flag should measure 3m by 2m and the area of the white cross must equal the area of the red surface. How wide are the stripes?

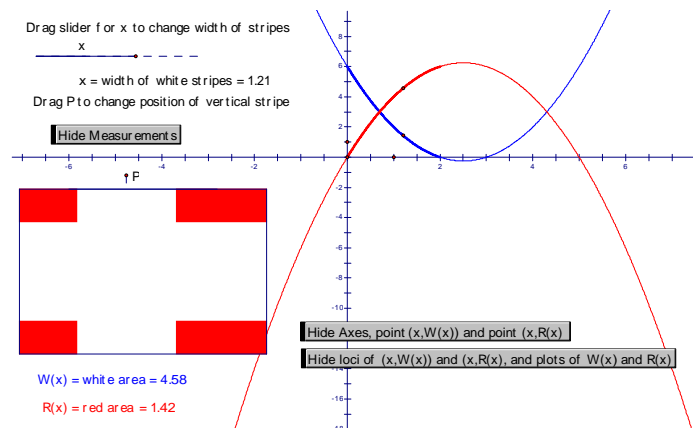


Figure 5. Sketch for the Danish Flag Problem

The Danish Flag Sketch should first be shown to students without the measurements for $W(x)$, the white area, and $R(x)$, the red area, the points $(x, W(x))$ and $(x, R(x))$, and the graphs. The hide/show buttons can be used to hide or show these objects. By dragging the slider for x , the width of the white stripes can be changed. Students should observe and comment on how this changes the white area and red area of the flag, and see how to make the flag all red or all white. They should then move the slider and “judge by eye” to find x when the two areas are the same. Next, the measurements for $W(x)$ and $R(x)$, the points $(x, W(x))$ and $(x, R(x))$, and finally the graphs should be displayed to give a better way to find when the areas are the same. Dragging the point P to change the position of the vertical stripe will help the students to see that two areas remain unchanged. These manipulations will help students to understand the problem and find an approximate solution.

Next the students should be asked to find the function rules for $W(x)$ and $R(x)$, $W(x) = 2x + 3x - x^2 = 5x - x^2$ and $R(x) = 6 - (5x - x^2) = 6 - 5x + x^2$. The students then solve by algebra the equation $R(x) = W(x)$, that is $x^2 - 5x + 6 = 5x - x^2$, for x . The algebraic solution found can be compared with that found by using the Sketch.

Shot on Goal Problem: What distance from the goal line should a rugby player kick to maximize the chance of converting a try? In other words, where should the player kick from so that the shot angle to the goal is a maximum?

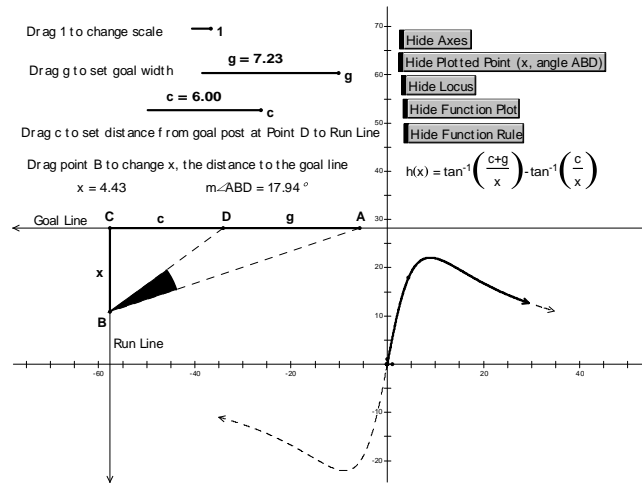


Figure 6. Sketch for Shot on Goal Problem

In the Sketch for Shot on Goal Problem only the problem diagram and the sliders to change the scale, goal width g , and distance c from the goal post at D to Run Line, are shown initially to students. The sliders are dragged to set the initial conditions of the problem. Then point B is dragged and the changes in the distance $BC = x$ and the $m\angle ABD$ angle are observed. To have students “see the relationship” between these two changeable quantities, the hide/show buttons are used to show the axes and then plot the point $(x, m\angle ABD)$.

Now the path of the point can be observed by students as B is dragged to change x . Students should describe how the point $(x, m\angle ABD)$ moves as x is changed before the shape of this path is revealed with the hide/show button for the locus of the point. This locus is part of a function which students should then be asked to find by making use of their knowledge of trigonometry. Students can check their function rule by plotting the rule to see if it gives the graph on which the locus lies.

As needed or desired, the correct function plot and its rule can be shown by use of the hide/show button for each. Students can observe how the graph of the function changes as the values for c or g are changed by using their respective sliders. The student can find a close approximation to the answer to the problem by computing and recording, on the Sketch, in their graphing or CAS calculator, the values of the function for x values near that needed for the maximum.

Conclusion

Dynamic geometry programs offer support for teaching much more than geometry. In this paper we have given examples that demonstrate three types of diagrams that can be constructed with these programs that support the teaching of algebra.

- Static, but colourful and sequenced, diagrams that can illustrate mathematical deductions step-by-step,

- Changeable diagrams that can show a generalization,

- Diagrams that can simulate a real situation.

In the examples for the last two types use was made of the software’s capabilities to calculate, plot points, produce a locus, and plot functions, with all the results linked to movement on the diagrams that produces changes in these results.

Although students can create their own diagrams, we believe that diagrams prepared in advance by teachers themselves or downloaded from resources are probably more useful for many teaching situations. The students can then focus on the mathematics at hand, manipulating the diagram or watching others do so, thereby understanding the problem better and being better able to take up the challenge of making use of algebra to solve it.

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