Chapter XX

FACILITATING MIDDLE SECONDARY MODELLING COMPETENCIES

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One method of engaging secondary students in mathematics classes is through taking a real world modelling approach to teaching, beginning in the middle years (Years 8/9) and developing students' modelling abilities into the senior secondary years. This chapter documents a case study in an Australian secondary school where such an approach is being established. The purpose is to provide an approach where Year 9 students experience a sense of success pivotal to their continued engagement with the modelling process. As the aim is that students have well developed modelling skills within a classroom context by Year 11, students' handling of transitions between phases of the modelling process are of particular interest. From intensive analysis of student responses whilst undertaking two extended tasks, a framework is developed for identifying student blockages during transitions.

1. BACKGROUND

This chapter reports aspects of a program in which mathematical modelling is being introduced to Year 9 students in a Victorian secondary college as the teacher concerned perceives this as a way to engage middle years students in their classroom mathematics. Primarily our approach is driven by the desire to obtain a mathematically productive outcome for a problem with genuine real-world motivation. Sometimes this is directly feasible, but at other times it is more a "life-like" problem that is the subject of modelling; however, the solution must take seriously the context outside the classroom within which the problem is located, in evaluating its appropriateness and value.

The world outside the classroom is a swampy place as far as problem contexts are concerned, where real data are usually messy and the mathematical methods needed to deal with them sometimes involve improvisation. To suggest that such an important part of reality should be "cleaned up" before a problem is presented is, we believe, to destroy a significant element of its authenticity. Consequently, we include the appropriate use of technology[1] as central to our purpose, and its integration with mathematics within the modelling process as creating essential

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[1] An exception is where technology is not available. Then the mathematics must be amenable by hand methods alone, and data provided accordingly.

challenges about which we need to know much more.

Data for this chapter have been generated within the RITEMATHS project, an Australian Research Council funded project of the Universities of Melbourne and the University of Ballarat with six schools and Texas Instruments as industry partners. The fourth author is leading initiatives in one of the schools. In summary, matters we are particularly interested in exploring are represented by intersections between mathematical content, technology, and modelling (Figure 1), and this chapter elaborates aspects of how we have gone about teaching and researching these interests.

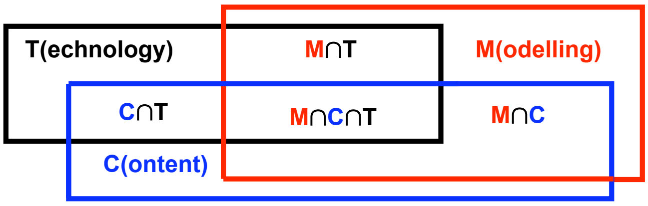


Figure 1. Interactions between modelling(M), mathematics content(C), and technology(T)

2. THE COMPONENTS

2.1 Mathematics Content

It has long been recognised that when learning to model it is unreasonable to expect students to access unaided, and apply mathematics that is at the frontiers of their experience and expertise or beyond, that is, problems need to be such that the mathematics required for solution is within the range of known and practised knowledge and techniques. It may not be clear however, just which mathematics is appropriate for the job at hand – such decisions *are* part of the requirements of the modelling process. This is a well-trodden area so we shall not elaborate further.

2.2 Technology

With respect to technology the situation is complicated, not only by the knowledge and facility required, but also by the skill and confidence with which students work with particular technologies. A *technology-rich teaching and learning environment* (TRTLE) affords new ways of engaging students in learning mathematics (Brown, 2005). The presence of electronic technologies in the classroom can fundamentally change how we think mathematically and what becomes privileged mathematical activity. Affordances are the offerings of such an environment for both facilitating and impeding learning, they are potential relationships between the teacher and/or student and the environment. Affordances need to be *perceived* and *acted* on by teachers and students alike in order to take advantage of the opportunities arising. This TRTLE includes opportunities for both teacher and students to enact a variety of affordances to support the learning of mathematics in ways well beyond those necessary simply for the production or checking of results. In the classroom reported here the teacher moves purposefully to integrate technology throughout the teaching and learning process with the goal of developing students' conceptual understanding and skills. In particular, the use of multiple representations, easily accessible with a graphing calculator, and tasks that are amenable to electronic technology, harness the opportunity for students to use technology to stimulate higher order thinking in the context of modelling real-world situations. Scaffolding of technical aspects is provided through demonstration or peer interaction when warranted, in an endeavour to ensure task cognitive demand is not increased by technology use (Stillman, Edwards, & Brown, 2004).

Another factor of importance in technology use is the level of confidence and facility that students believe they possess in accessing and using technology. The existence of qualitatively different levels of confidence and expertise (see Goos, Galbraith, Renshaw, & Geiger, 2003) clearly has relevance for an introduction to mathematical modelling with technology at the middle years of secondary school. Students in the class studied came with a variety of prior experiences of scientific calculators and spreadsheets from previous classes. Year 9 was the first year that students were required to have their own laptop and TI-83 Plus graphing calculator, and the first time they used graphing calculators.

2.3 The modelling process

Various versions of the diagram in Figure 2 exist (e.g., Edwards & Hamson, 1996) and most may be recognised as descendants from the one originally used by the Open University (UK). It represents both a description of a dynamic iterative process, and a scaffolding infrastructure to help beginning modellers through stages of what can initially appear to be a demanding and unfamiliar approach to problem solving. Some such framework is desirable to support a modelling initiative, whether overtly as in Figure 2 or as a means to structure effective sequences of activities, both mathematical and pedagogical. The arrows represent fundamental transitions that depict the dynamic nature of the process, and these are associated with some of the most demanding phases of the modelling process – particularly we would suggest the specification of a solvable mathematical problem from a messy real world context, and its formulation in a way that will lead to an appropriate mathematical solution. In this chapter we are interested particularly in these and other transitions between phases as students engage with their first experiences of modelling activity.

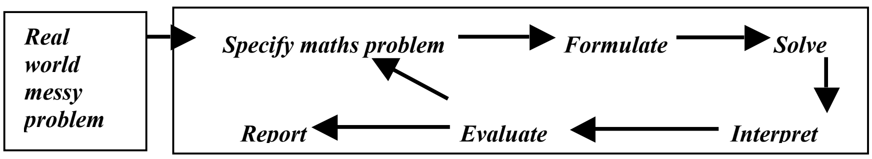


Figure 2. Modelling Process Chart

The research being undertaken is part of a design experiment (Collins, Joseph, & Bielaczyc, 2004), which was beginning its second cycle at the time of data collection. The research focus has been fashioned by the nature of the program, and beliefs about how central elements of interest are best captured and analysed. A classroom in which mathematical modelling is being enacted is a varied and unpredictable place. There is intense activity, fallow time when impasses emerge, and spontaneous and unforeseen actions by students engaging with new material and challenges. Such occurrences trigger at times unplanned interventions by teachers, who grasp moments they could not themselves have envisaged, to capture, extend or clarify learning opportunities that have suddenly emerged. Such a culture is central to the process of teaching mathematical modelling skills, where successive implementations even by the same teacher vary substantially in their detail. This is to be celebrated even though it renders efforts to conduct controlled experiments highly questionable. Hence our focus is located elsewhere – at the level of individuals learning and applying modelling skills in a TRTLE. In particular we aim to learn more about the critical points that represent transitions between phases in the solution process. These, represented by arrows in Figure 2, refer respectively to movements from:

1. Real world problem statement → Specification of a mathematical problem;
2. Mathematical problem statement → Formulation of an approach to solution;
3. Formulated approach → Complete solution of mathematics;
4. Mathematical solution → Interpretation in the context of the model;
5. Consequences of interpretation → Evaluation of the model quality;
6. Evaluation of model → Second or higher review and refinement of model;
7. Evaluation of model → Production of a report and recommendations.

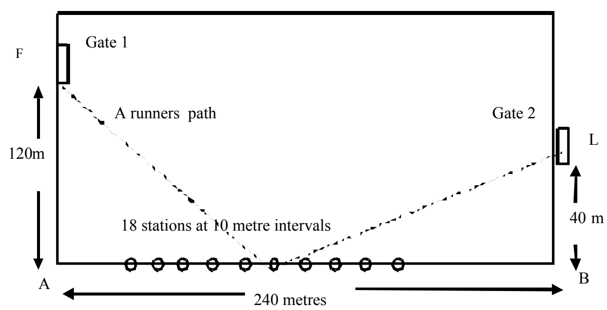
We focus on 1 - 4 specifically, noting that 1 and 2 have long been identified as among the most difficult phases of the entire process, acting as gatekeepers to problem access. We therefore examine in detail how students approach and perform in these areas of transition, while learning modelling in an environment characterised by the interactions portrayed in Figure 1.

3. THE PROBLEMS

Below we summarise aspects of two problems (Figures 3 & 4) that have formed a focus for the introduction of modelling to a class of 28 Year 9 students (11 male, 17 female, 14-15 year olds) at an independent school in Melbourne.

3.1 Orienteering - Cunning Running (CR)

In the Annual "KING OF THE COLLEGE" Orienteering event, competitors are asked to choose a course that will allow them to *run the shortest possible distance,* while *visiting a prescribed number of check point stations.* In one stage of the race, the runners enter the top gate of a field, and leave by the bottom gate. During the race across the field**,** *they must go to one of the stations* on the bottom fence. Runners claim a station by reaching there first. They remove the ribbon on the station to say it has been used, and other runners need to go elsewhere. There are 18 stations along the fence line at 10 metre intervals, the station closest to Corner A is 50 metres from Corner A, and the distances of the gates from the fence with the stations are marked on the map.



THE TASK

Investigate the changes in the total path length travelled as a runner goes from gate 1 to gate 2 after visiting one of the checkpoint stations. To which station would the runner travel, if they wished to travel the shortest path length?
For the station on the base line closest to Corner A calculate the total path length for the runner going Gate 1 – Station 1 – Gate 2. Use Lists in your calculator to find the total distance across the field as 18 runners in the event go to one of the stations, and draw a graph that shows how the total distance run changes as you travel to the different stations.
Observe the graph, then answer these questions. Where is the station that has the shortest total run distance? Could a 19th station be entered into the base line to achieve a smaller total run distance? Where would the position of the 19th station be? If you were the sixth runner to reach Gate 1, to which station would you probably need to travel? What is the algebraic equation that represents the graph pattern? Plot this graph of this equation on your graph of the points. If you could put in a 19th station where would you put the station, and why?
(Additional suggestions were provided as to how the work might be set out, and for intermediate calculations that would provide some task scaffolding.)

Figure 3. *Cunning Running* Task

The solution involves the calculation of total path as the sum of two segments, followed by graphing, construction of an algebraic model, verification, interpretation, and the search for a nineteenth station optimally located. Figure 4 shows a typical graph produced by students who chose to use a spreadsheet.

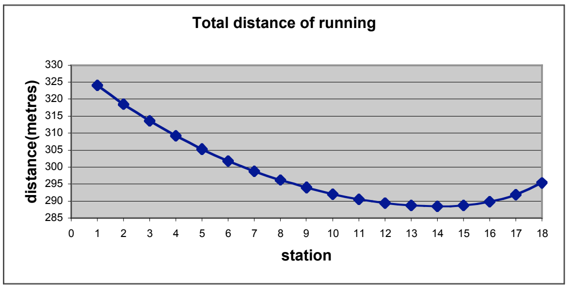


Figure 4. Typical spreadsheet graph produced in solution to *Cunning Running*

3.2 Shot on Goal (SOG)

Many ball games have the task of putting a ball between goal posts. The shot on the goal has only a narrow angle in which to travel if it is to score a goal. In field hockey or soccer when a player is running along a particular line (a run line parallel to the side line) the angle appears to change with the distance from the goal line. At what position on a run line, does the player have the widest opening for the shot on goal?
Assume you are not running in the GOAL-to-GOAL corridor. Find the position for the maximum goal opening if the run line is a given distance from the side line As the RUN LINE moves closer or further from the side line, how does the location of the position for the widest view of the goal change? (Relevant dimensions given).

Note: Because it was felt to be more inclusive in a co-educational school, and because a number of students play the game, hockey was selected to provide the specific context. There are some caveats associated with this decision. Firstly a shot on goal in hockey is only allowed from a point within the “almost” semi-circular penalty area, and hence technically only some of the run lines are feasible. This aspect can be included at a later stage by first finding the location of the best shooting position in terms of angle as is required by the question, and then checking its position relative to the penalty area. (With Soccer there are no such restrictions).

Figure 5. *Shot on Goal* Task

Figure 6 shows calculations obtained using LISTs of a TI-83 Plus graphing calculator, for positions of the goal shooter at perpendicular distances from the goal line of between 14 and 20 metres along a typical run line that is 10 metres from the sideline. (Width of goalmouth is 3.67 metres). The maximum angle is highlighted at L4(17). Further accuracy can be obtained by testing nearby values, and a graph can be drawn. In practice a player often uses a zigzag run. Discussion can be used to infer that whatever the path the ultimate shot is from a position on some run-line.

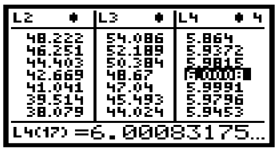


Figure 6. Kim’s LISTs for a run line 10m from the side line.
Note: Actual formulae used: L2 = $\tan^{-1}$(15.67/L1), L3 = $\tan^{-1}$(19.33/L1), and L4 = L3 - L2.

3.3 Interactions between modelling, technology, & mathematical content

Figure 7 illustrates some of the interactions that students have to cope with between the modelling process (M), mathematical content (C), and technology use (T) as foreshadowed in Figure 1 during the solution of these tasks.

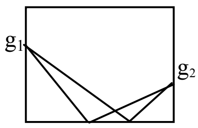
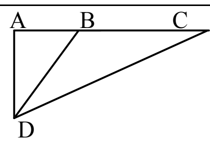
Interactions	*Cunning Running*	*Shot on Goal*
M∩C	Represent problem. To minimise sum of distances (Pythag) 	Represent problem. To maximise ∠BDC 
M∩T	Strategy decisions: e.g. choice of GC functions, Excel.	Strategy decisions: e.g. choice of GC functions, Excel.
C∩T	Computation of distances, plotting points, interpreting graphs, verification of algebraic model graphically.	Computations involving tangent, Use of inverse tan to find angles for different run lines, and different positions on run lines.
M∩C∩T	Carrying through to solution: Formulate →calculate paths → plot graphs→algebraic model→ interpret output→determine implications in terms of problem requirement.	Carrying through to solution: Choice of approach → appropriate diagram → calculations of angles→ correct graphical output→ interpretation → implications.

Figure 7. Interactions between modelling process, technology, and mathematics

4. TASK IMPLEMENTATION

The two tasks, *Cunning Running* and *Shot on Goal,* were the vehicles used to generate intensive data; collected by means of student scripts (24 and 28 respectively), videotaping of teacher and selected students, video and audiotaped records of small group collaborative activity, and selected post-task interviews (8 and 4 respectively). We seek to identify and document characteristic levels of performance; the occurrence or removal of blockages; the respective use of numerical, graphical, and algebraic approaches; quality of argumentation; and the respective interactions between modelling, mathematical content, and technology. The purpose is to enable an analysis of issues and activities impacting on transitions 1 - 4 identified above, with implications for both the learning and teaching of introductory mathematical modelling in the middle secondary years.

The first modelling task, *Cunning Running*, was undertaken in the fourth week of the school year. Previous learning experiences included the students' first introduction to LIST operations on a graphing calculator, where students worked as a class through the solution of a contextualised task. In this task numerical, graphical, and algebraic methods were used. *Shot on Goal* came two months after the first task but there had not been a focus on graphing calculator use in the intervening time. Both tasks were undertaken over about a week of class time.

5. EXAMINING TRANSITIONS

As indicated above we are interested in transitions between phases of the modelling process, and in identifying *blockages* that impede students in moving between these. In *Cunning Running* students were required to vary distances with the purpose of minimising the total distance run, while *Shot on Goal* involved looking at varying angles with the aim being to maximise the shot angle. Surface similarities of these tasks tend to obscure different levels of complexity in the respective formulations.

5.1 Real world problem statement to mathematical problem specification

The challenge here is in identifying the *key elements* that will form the focus for model building. This involves:

- *Firstly deciding the nature of the element (what kind of mathematical entity)*
- *Secondly identifying the specific element(s) to be focused on.*

The element was a 'distance' in *Cunning Running*, and an 'angle' in *Shot on Goal*. Both these were compound entities that needed to be constructed from other components present in the real situation (line segments or angles). Blockages occurred, particularly in the second problem, and the blockage was only unlocked for some students, when the teacher provided a supporting physical demonstration in which class members either watched or participated. This was followed by a debate by selected students using diagrams on the whiteboard about which angle was the focus. The physical enactment of the situation proved crucial for some students. "It helped to explain like what we hoped to find out. Like I didn't really get what we were trying to do. And that kind of explained what angles we were trying to find."

While *interpretation* is an acknowledged component of the modelling process it is usually associated with giving meaning to a solution in its real world context. Interpretation in a different sense is seen here to be central in specifying a mathematical problem in the first place. No effort needs to be spared in ensuring that students receive a thorough appreciation of a problem context. This may involve direct experience, film or video, simulation, discussion and diagrammatic representation, or written description. Assuming that a mental representation will suffice is demonstrably suspect, and indeed arguably inconsistent with the purpose of solving problems grounded in real situations.

5.2 Mathematical problem to formulation of a model

Key decisions with the potential to generate blockages in this transition were identified along the following dimensions:

•*Deciding how to represent an identified element*, so that known mathematical formulae could be applied (e.g., in SOG how to find the angle that had been identified – two different decomposition methods were available).

•*Choosing to use technology to establish a calculation path* (e.g., in SOG inverse tan calculations provided an efficient means of obtaining the required angle - provided that the mathematics within the approach was understood).

•*Choosing to use technology to automate extension of application of formulae to multiple cases* (e.g., CR & SOG graphing calculator LISTs/spreadsheet). This required both technical facility and a strong grasp of the modelling requirements of the moment - recognising that generalisation via this activity was needed.

•*Recognising only one dependent and one independent variable is to be specified in an algebraic model* (e.g., in CR several students used algebraic models with several variables such as $x + y = \sqrt{(120^2 + d^2)} + \sqrt{(40^2 + d2^2)}$).

•*Recognising that a particular independent variable must remain uniquely defined* throughout an application (e.g., in CR conflict occurred when '*x'* was defined as the distance from a station to corner A then from corner B in different parts of a formula).

•*Recognising that a graph can be used on function graphers but not data plotters to verify an algebraic equation* (e.g., CR verification method works on graphing calculator but not on a spreadsheet).

•*Recognising when additional interim results are needed to enable progress* (e.g., in attempting to place the 19th station in CR).

•*Introducing flawed problem data* into the formulation phase (e.g., two students took a stepped trajectory towards the goal instead of advancing down a specified run line in 1 m intervals).

•*Selecting appropriate procedures when alternatives exist* (e.g., in CR most students used an EXCEL joined scatterplot by joining numerical data generated by their graphing calculator LISTs; however a minority used formulae to generate the data - a fundamentally different approach).

5.3 From formulated approach to solution of mathematics

As might be expected the interaction between mathematics and technology featured strongly in this transition.

•*Problems that occur as logical consequences of earlier errors in formulation* (e.g., approaches giving different meanings to the same symbol in different parts of CR).

•*Problems in applying formulae correctly* (e.g., in SOG using inverse tan successfully to find angles).

•*Using technology to automate extensions of application of formulae* to multiple cases (e.g., CR & SOG: graphing calculator LISTs/spreadsheet).

•*Using technology to produce a graphical representation* (e.g., CR & SOG: graphing calculator StatPlot, function graph or spreadsheet chart).

•*Applying algebraic simplification processes to symbolic formulae.* (e.g., in CR & SOG, concatenation of graphing calculator LIST formulae to produce a function).
•*Applying the rules of notational syntax accurately.*
•*Using technology to verify an algebraic model* (e.g., CR & SOG: by producing a graph, or by substitution into a formula on the homescreen, or by entering a specific functional definition on a graphing calculator).
•*Reconciling unexpected interim results with real situation* (e.g., in SOG one student incorrectly expected that the angle would continue to increase as the player advanced along the run line. He did not accept that his correct calculations were valid, and after debating with other students, was convinced only by another physical simulation of the activity).

5.4 From mathematical solution to interpretation within the model context.

The following illustrate where student attempts at interpretation point to the occurrence of blockages, potential or actual:
•*Routine interpretations varied in depth from bald statements to integrated explanations* (e.g., when asked in CR "Does running via station 1, or 2, or 3 make any difference to the overall length of the run?", responses ranged from bald assertions such as, "It makes a difference," to integrated arguments such as "Yes, it does the closer you are to corner A, the further the distance you have to run.")
•*Interpretations in which the required outcome is amended to a variation introduced as the consequence of a preferred method of approach* (e.g., in finding the location of the station with shortest total run distance in CR, most students used their numerical lists to identify a particular station, rather than their graphs, which more legitimately provided the location of a minimum. The interpretation then referred to a station number rather than a location. While this did not matter here as the minimum was actually at station 14, it would be significant if the question was set so that the required station was in a position between others).
•*Differences in precision when numerical values are important* (e.g., in describing the optimum position for a station in CR, most gave the station number they considered to involve minimum distance. One student added that it was "3/4 between gate 1 and gate 2", while another, seeing no need for any mathematical calculations at all simply asserted it was "towards the end").
•*Tensions in deciding how knowledge of mathematics would actually impact on the real situation.* (e.g., in CR – a number of students gave credit to the sixth runner as a mathematician who would know which station to run to – others said that in the heat of a race you wouldn't really worry about distance, and just head for the 6th station).
•*Choosing criteria to use in a later aspect of a task that requires interpretation to establish a method of approach* (e.g., in CR placement of the 19th station saw the use of various criteria – rather than applying the distance criterion [correct approach], some said it should follow 18th; some just put it 10 m from first or last station; while one student ignored all constraints, and put it on the straight line joining the gates).

6. REFLECTIONS

Analysis of student responses during two modelling tasks has focused on the transitions between phases in the modelling process. For each transition we have been able to identify generic entries that document possible blockages halting or slowing the progress of beginning modellers. These have been illustrated by specific examples from the first such tasks implemented at the school in question. Once a mathematical approach has been formulated, students need to transform a viable approach into a solution. Our analysis has shown it is necessary to carry out mathematical and technological activities successfully to do this and the framework has been useful in identifying what prevents this happening - lack of mathematical knowledge general or specific, lack of technological skills, technical expertise, or accuracy. The complexity of interactions between modelling, mathematical content, and technology when solving modelling problems such as these in a TRTLE emphasise the importance of student perception and judicious enactment of affordances offered by such an environment. Having arrived at a mathematical solution, there are further potential blockages as students seek to make meaning of this in contextual terms - keeping in mind that the interpretation may be simple direct translation, or more demanding. The framework reveals the factors that give rise to the hierarchies of quality found in interpretation and the resultant blockages come from mistakes in interpretation, overlooking interpretation, providing superficial or irrelevant responses, and not seeing where interpretation is required for further progress to be made.

The framework has potential for identifying and documenting specifically modelling competencies with which beginning modellers need to have facility in order to successfully apply the mathematics they know in real problem settings. It also has potential as a research tool for analysing student activities during solution of modelling tasks. These considerations will guide future research activity, as students engage with further problems. The intention is to search for a higher synthesis, using additional data to clarify and illustrate conceptual structures indicative of the transitional issues provisionally identified in this chapter.

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