SHOT ON GOAL

Name: ________________________________

introducing
TI-nspire CAS

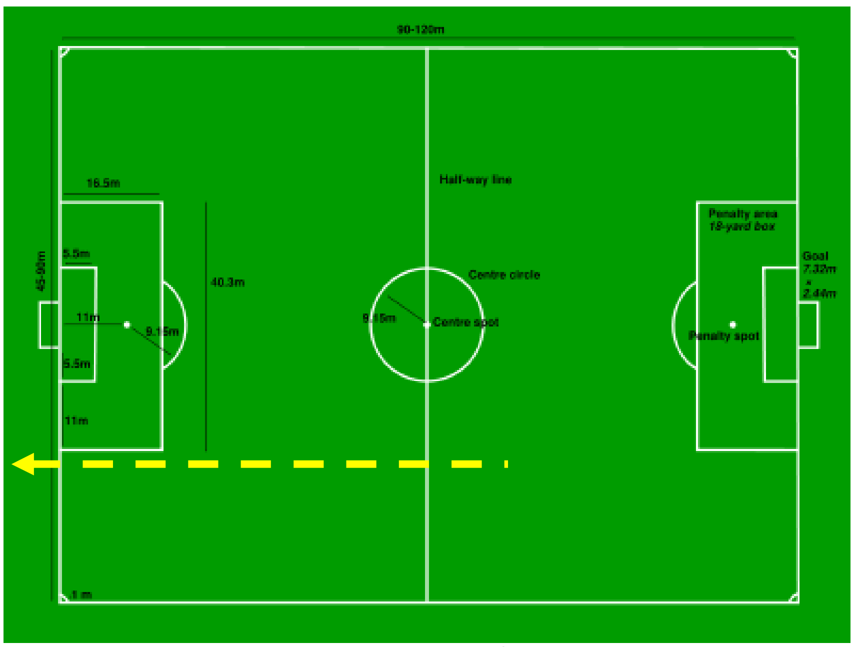
Football – scoring a goal and trigonometry

Teachers Teaching with Technology

THE TASKS

You are an assistant coach with the local football team, the Luther Lions FC. At a training session, your task is to demonstrate to the young players the zones on the football pitch which show the greatest opening between the goal posts. This knowledge will assist forwards in knowing when to shoot as the angle of the goal opening is greatest and the defenders in revealing which parts of the ground need to be controlled.

VISUALIZE THE SITUATION



(The run line for the player is the dashed line)

There is only 5 minutes remaining on the clock until the end of the game. Your team has the ball. Jonathan Angle, your star forward, makes a run (dotted line). He has broken clear from the defenders. His path is on a line parallel to the side touch lines and 15m to the left of the near post. He unleashes a powerful left-foot shot on goal.

THE QUESTION

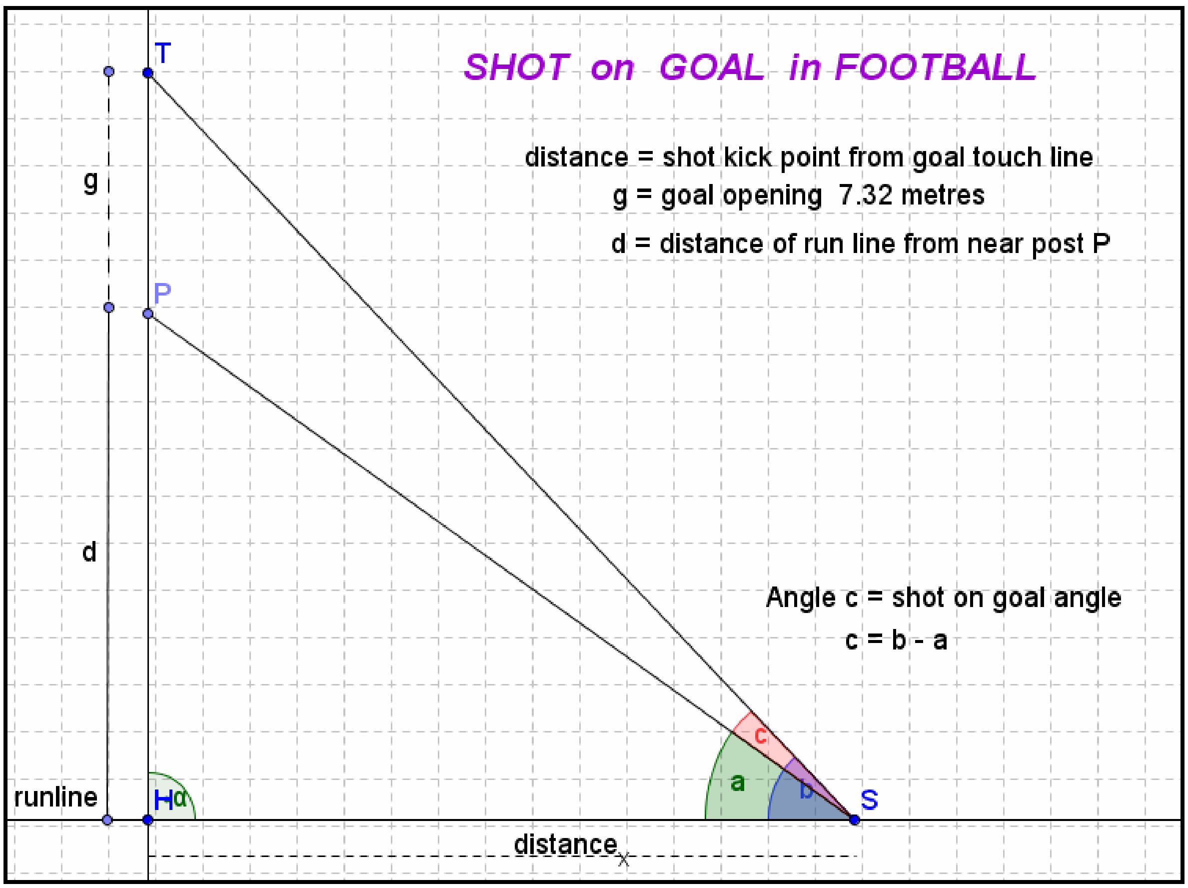
Has Jonathan kicked the ball from the position giving the maximum angle for opening up the goal thereby increasing the chance for scoring a goal?

PART 1

1.1 Clarifying the Idea – Drawing the Diagram

A football game is a 3-D situation. You require a 2-D dimensioned PLAN of the field to show the mathematical relationships of the angle from the kicking spot to the goal posts.

1. Your diagram should be completed on the GRAPH PAPER below.
2. The football field shows the positions of the **Goal Line**, the **Touch Line**, and the **Run Line.**
3. Mark a **Spot** from which the shot could have been taken and the **Angle formed** with the goal.
4. On your diagram clearly mark all dimensions. Mark the right angles, and angles required for the calculation of the shot on goal angle. The vertices need to be labelled (A, B,C …) to allow you to refer to lines and angles in your work.



1.2 Clarifying the Idea – Determining the Angle

You are to calculate the size of the angle formed from the kicking spot to the near and far posts of the goals.

1. Record the position of 4 different kicking spots on the Run Line which is 15m to the left of the goal posts.

The distance (x) of 4 points on the run line from the goal touch line

S_1 x_1 = 9 metres

S_2 x_2 = 12 metres

S_3 x_3 = 15 metres

S_4 x_4 = 20 metres

2. Find the Angle of the Shot from each of these kicking spots. (Four fully worked manual calculations to determine the angle of the shot on goal are required.)

C = angle of shot on goal

X = distance along the run line of the kick spot

D = distance run line is offset from the near goal post D = 12 metres

G = Goal post width G = 7.32

$$\llcorner C = \llcorner B - \llcorner A$$

Problem 1

X = 9 m

$\text{Tan b} = \frac{22.32}{9}$ $\llcorner b = 68.04^{o}$

$\text{Tan a} = \frac{15}{9}$ $\llcorner a = 59.03^{o}$

Angle c = 9.01^{o}

Problem 2

X = 12 m

$\text{Tan b} = \frac{22.32}{12}$ $\llcorner b = 61.76^{o}$

$\text{Tan a} = \frac{15}{12}$ $\llcorner c = 51.34^{o}$

Angle c = 10.42^{o}

Problem 3

X = 15 m

$\text{Tan b} = \frac{22.32}{15}$ $\llcorner b = 56.10^{o}$

$\text{Tan a} = \frac{15}{15}$ $\llcorner a = 45^{o}$

Angle c = 11.1^{o}

Problem 4

X = 20 m

$\text{Tan b} = \frac{22.32}{20}$ $\llcorner b = 48.14^{o}$

$\text{Tan a} = \frac{15}{20}$ $\llcorner c = 36.87^{o}$

Angle c = 11.27^{o}

1.3 Clarifying the Idea – Thinking about the Problem

How do the results of the calculations suggest that there is a point for a best shot on goal ?

There is an indication that a maximum or at least an upper limit for the size of the angle of the shot of goal may exist as the results (9, 9.01) ; (12, 10.42) ; (15, 11.1) and (20, 11.27) indicate a decrease in the average rate of change of the angle increase with increase in distance from the goal touch line [0.47 °/m to 0.05 °/m].

PART 2

2.1 Generalize the Situation – Capturing new data

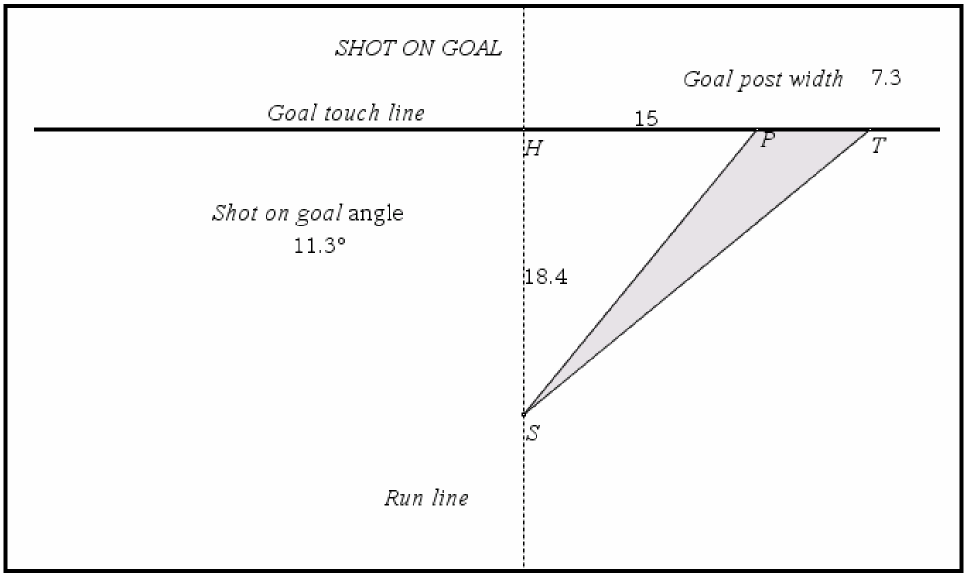
Using technology to replicate the manual steps

You have done 4 manual calculations for the values of the angle for the shot on goal. We now will capture more information to illuminate how the values of angle for the shot on goal change as you approach the goal touch line.

CONSTRUCTION OF A DYNAMIC GEOMETRY APPLICATION

Open a Graph and Geometry application

1. Set the scales on the axis to cover at least one quadrant of the football field
2. Hide the scales, but reveal the grid. On this grid, construct a dynamic geometry application of the shot on goal. The application should
   a. Show the goal touch line with the goal posts P and T. The distance between P and T is 7.3 metres.
   b. 15 metres to the left of left post, construct the run line SH, where S is the position from where the shot is taken.
   c. The ∟PST is the shot on goal angle.
   d. Measure the distance SH and the ∟PST.
   e. Open the variable menu, and link the variable **'d'** to the distance SH and the variable **'s'** for ∟PST.



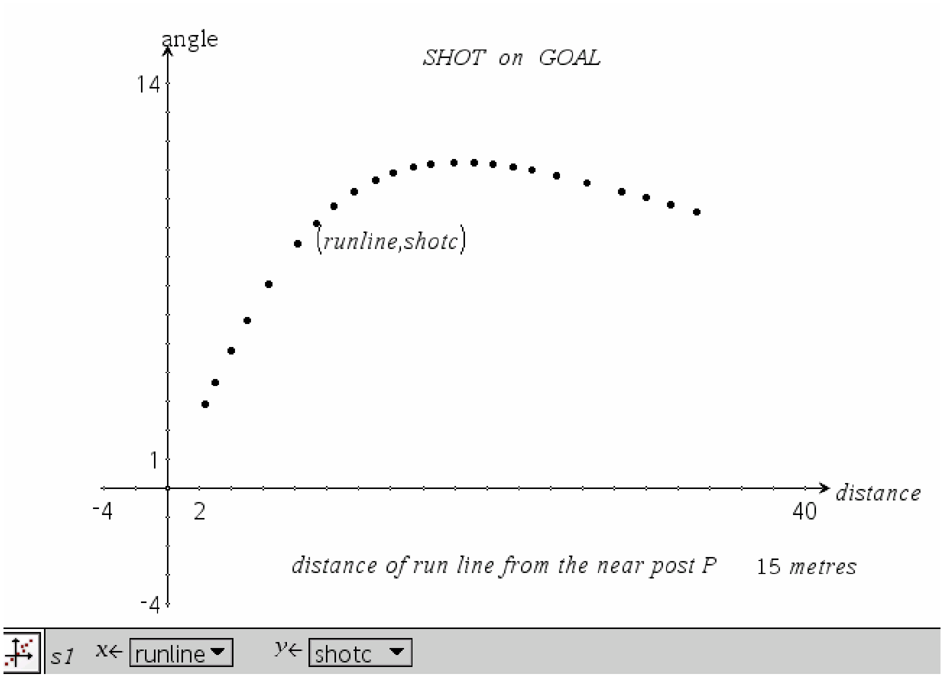
Open a List and Spreadsheet application

3. Set up the manual capture equation the values for the distance along run line and the angle for the shot on goal .
4. Capture a record of the 30 values for the angle of the shot on goal and the distance from the Goal touch line to the kicking spot along the run line.

	A runline	B shotc	C	D	E	F	G
◆	=capture(x,0)	=capture(pst,0)					
1	2.36363636364	2.89709833427					
2	3	3.63871099068					
3	4	4.75009717152					
4	5	5.78239232224					
5	6.36363636364	7.043320901					
6	8.18181818182	8.43980625393					
7	9.36363636364	9.17187719934					
8	10.4545454546	9.7305427137					
9	11.7272727273	10.2507643503					
10	13.0909090909	10.6669900429					
11	14.1818181818	10.9076751241					
12	15.3636363636	11.0883909627					
13	16.4545454546	11.1915485758					
14	17.9090909091	11.2495669769					
15	19.1818181818	11.2390928649					
16	20.3636363636	11.1881445268					
17	21.6363636364	11.0976237774					

Open a Graph and Geometry application

5. Draw a scatter plot of the shot on goal angle (y) as a function of the distance along the run line (x).



2.2 Generalize the Situation – Interpreting the Data

Searching for the maximum goal opening

a) Trace along the points plotted on the graph. Determine the highest value for the shot on goal angle. What is the value? What distance is the kick from the goal touch line? At about 18 metres the shot on goal angle is about 11.2°

b) Examine on your table. Between what two kicking spot distances do you conclude that the maximum angle for the shot on goal must happen?

10	13.0909090909	10.6669900429
11	14.1818181818	10.9076751241
12	15.3636363636	11.0883909627
13	16.4545454546	11.1915485758
14	17.9090909091	11.2495669769
15	19.1818181818	11.2390928649
16	20.3636363636	11.1881445268

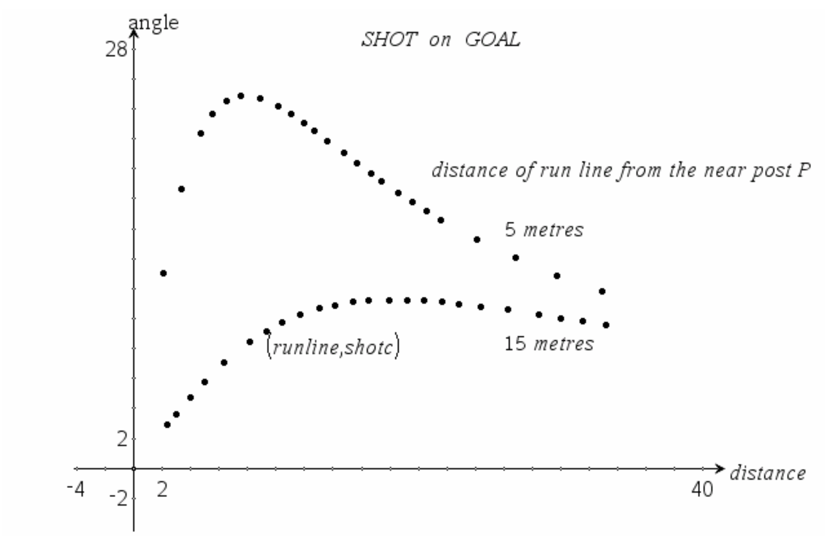
The maximum angle for a shot on goal would be about 11.3° and it occurs between 16.5 m and 19.2 m from the goal touch line along the 15 m run line.

c) If you change the distance the run line is from the near post (from 15 metres to 5 metres), what effect do you think there would be on the shot on goal angle? If you were a defender how would employ this knowledge in marking an opposition player with the ball?

Capture data for the angle on a 5m run line to investigate these questions.

1. At all positions along the 5 metre run line the angle for the shot on goal is greater than the corresponding angle of the shot on goal for the 15 metre run line.
2. The maximum angle for the shot on goal is closer to the goal for the 5 m run line than on the 15 m run line.
3. The maximum angle for the shot on goal on the 5 m run line appears to be more than double than the angle on the 15 m run line.
4. The distances between which the angle is near maximum is smaller for the 5 m run line than for the 15 m run line.

As a defender you would want to force the attacking player wider, towards the touch line, but be aware that the opportunity to shoot with the maximum opening of the goal line is greater wider out, but it is further away from the goal line. You would need to be careful of crosses made out wide that are passed forward – generally these will increase the opening for the shot. Likewise there are zones close to the post that must be blocked when a corner is taken.



2 .3 Generalize the Situation – Modelling the Data

Your task is to find an algebraic model to fit your graph and calculations. One method of discovering an algebraic model is to generalise what you did when you did manual calculations.

A. A clear idea of the purpose of the investigation

1. What is it that I am trying to discover?

How does the angle for the shot on goal changes as the distance from the touch line increases? The angle size is a function of distances (along the run line and position from the near post.)

B. A knowledge of the relationships

1. What factors control the changes in the value I am trying to discover?

The goal width (G) is fixed at 7.32 metres. The distance (d) from near post changes is a parameter. The distance (X) affects the angle size

2. What values remain constant in the investigation?

The goal width (G) is fixed at 7.32 metres.

3. What mathematical ideas / procedure / rules show how these values are related? Are the values directly related, or do you need to find transitional values?

The trigonometric ratios relate lengths to angles. Unknown lengths can be calculated using the Cosine Rule (or Pythagoras' Theorem in Right angle triangles). In this situation, there are right angle triangles with the value for the Tangent (angle) relating the lengths given.

THE EQUATION FOR THE EVENT

What is the algebraic equation that represents pattern revealed in the graph?

Hints

1. Observe what you did in the manual calculations.
2. Imitate the steps of the solution of problem using variables not numbers
   - Keep a list of the variable names and what they represent.
   - Some variables have values that do not alter in any situation. Do you have any variable of this type? If you do, then you can replace the variable with this value.
   - Some variables change only in different scenarios. Do you have any of these types of variables.
   - Other variables change continually throughout the investigation.
3. The final equation is a statement of what you are trying to find, using the values that are causing the observed changes.

Angle relationship

∟c = ∟b - ∟a

Trigonometric ratio

Tan(c) = tan (b) – tan (a)

$$c = \left(\tan^{-1}\left(\frac{22.32}{x}\right) - \tan^{-1}\left(\frac{15}{x}\right)\right)$$

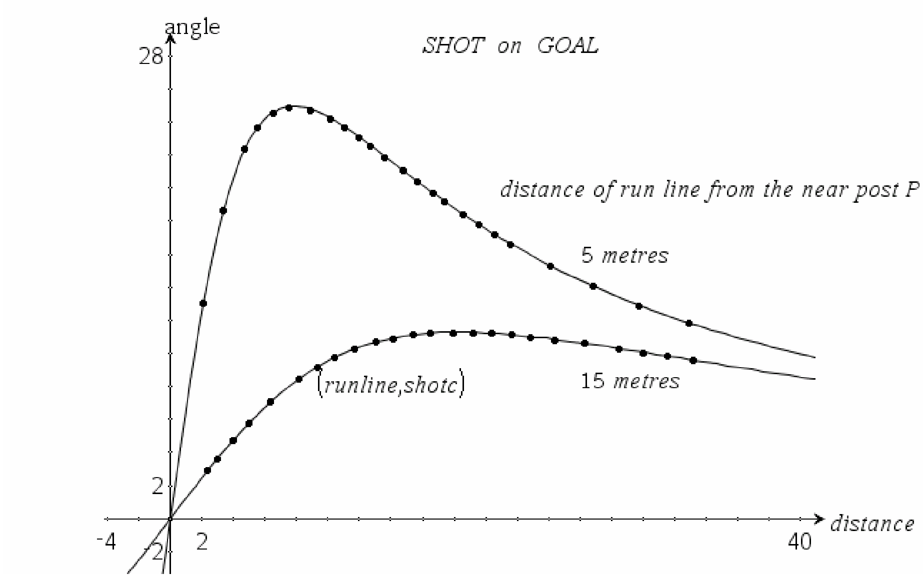
$$c = \tan^{-1}(\tan(b-a)) \mid \tan(b) = \frac{22.32}{x} \text{ and } \tan(a) = \frac{15}{x}$$

$$c = \tan^{-1}\left(\frac{7.32x}{x^2 + 334.8}\right)$$

3 .1 New insights – Checking the Model

Draw the graph of your mathematical model on the scatter plot of the distance – angle size values.

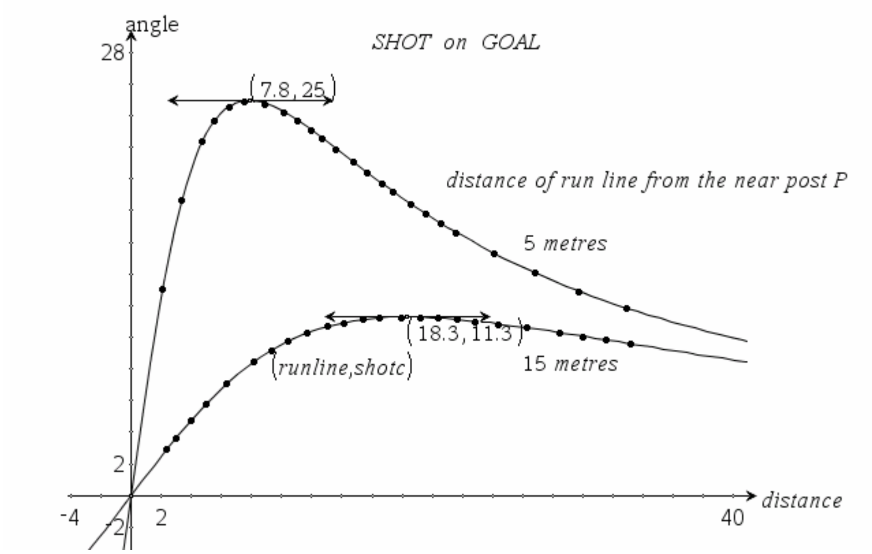
1. How does the graph of your model confirm / reject your equation?



The points captured lie on the equation of the curves, hence they validate the algebraic insights used in constructing the equations.

2. What is the exact position on the field where the maximum angle for the shot on goal occurs for your run line? Explain how you worked out the value of this maximum angle.

Construct the tangent line to the curve, and find the maximum value [M} when the gradient of the tangent line is zero. Confirm this algebraically by differentiating the equations of the curves, and finding the x value for when the gradient function is zero.



3 .2 New insights – Checking the Model

Investigate how the position for the kicking spot of the maximum angle of the shot on goal varies as the run line moves nearer to the near goal post.

This investigation can be conducted either

1. EXPERIMENTALLY and GRAPHICALLY.

This will require you to collect data (using the applet) and plotting results (run line distance from post and distance to spot on run line) to reveal any the patterns in placement of spots for the maximum angle. Then, establish the equation.

2. EXTENDING THE ALGEBRAIC PROCEDURES.

This will require you to introduce a variable for distance of the run line is from the near post. Then, establish the equation using the power of CAS to handle the mathematical computations for the analysis.

Define $c=b-a$	*Done*
$\tan(c)$	$-\tan(a-b)$
tExpand$(\tan(b-a))$	$\dfrac{\cos(a)\cdot\sin(b)-\sin(a)\cdot\cos(b)}{\cos(a)\cdot\cos(b)+\sin(a)\cdot\sin(b)}$
$\dfrac{\tan(b)-\tan(a)}{1+\tan(a)\cdot\tan(b)}$	$\dfrac{\cos(a)\cdot\sin(b)-\sin(a)\cdot\cos(b)}{\cos(a)\cdot\cos(b)+\sin(a)\cdot\sin(b)}$
$\tan(b)=\dfrac{7.32+d}{x}$	$\tan(b)=\dfrac{d+7.32}{x}$
$\tan(a)=\dfrac{d}{x}$	$\tan(a)=\dfrac{d}{x}$
$\dfrac{\tan(b)-\tan(a)}{1+\tan(a)\cdot\tan(b)}\Big\vert\tan(b)=\dfrac{d+7.32}{x}$ and $\tan(a)=\dfrac{d}{x}$	$\dfrac{7.32\cdot x}{x^2+d\cdot(d+7.32)}$
$\dfrac{7.32\cdot x}{x^2+d\cdot(d+7.32)}\Big\vert d=15$	$\dfrac{7.32\cdot x}{x^2+334.8}$
Define $fc(x)=\tan^{-1}\left(\dfrac{7.32\cdot x}{x^2+d\cdot(d+7.32)}\right)$	*Done*

$\dfrac{\tan(b)-\tan(a)}{1+\tan(a)\cdot\tan(b)}\Big\vert\tan(b)=\dfrac{d+7.32}{x}$ and $\tan(a)=\dfrac{d}{x}$	$\dfrac{7.32\cdot x}{x^2+d\cdot(d+7.32)}$
$\dfrac{7.32\cdot x}{x^2+d\cdot(d+7.32)}\Big\vert d=15$	$\dfrac{7.32\cdot x}{x^2+334.8}$
Define $fc(x)=\tan^{-1}\left(\dfrac{7.32\cdot x}{x^2+d\cdot(d+7.32)}\right)$	*Done*
$\dfrac{d}{dx}(fc(x))$	$\dfrac{-419.405106036\cdot(x^2-d\cdot(d+7.32))}{x^4+2.\cdot(d^2+7.32\cdot d+26.7912)\cdot x^2+d^2\cdot(d+7.32)^2}$

The establishing of the equation for the shot on goal equation fc(x) and the derived equation to determine the position for the maximum angle for the opening for the shot on goal as a function of the position of the run line in relation to the near post (d) and the position for the kick (x).

$$\text{solve}\left(\frac{-419.40510603576\cdot\left(x^2-d\cdot(d+7.32)\right)}{x^4+2\cdot\left(d^2+7.32\cdot d+26.7912\right)\cdot x^2+d^2\cdot(d+7.32)^2}=0,x\right)\Big|d=25$$

$x=-28.4253408071$ or $x=28.4253408071$

$$\text{solve}\left(\frac{-419.40510603576\cdot\left(x^2-d\cdot(d+7.32)\right)}{x^4+2\cdot\left(d^2+7.32\cdot d+26.7912\right)\cdot x^2+d^2\cdot(d+7.32)^2}=0,x\right)\Big|d=30$$

$x=-33.4604243846$ or $x=33.4604243846$

$$\text{solve}\left(\frac{-419.40510603576\cdot\left(x^2-d\cdot(d+7.32)\right)}{x^4+2\cdot\left(d^2+7.32\cdot d+26.7912\right)\cdot x^2+d^2\cdot(d+7.32)^2}=0,x\right)\Big|d=20$$

$x=-23.3752005339$ or $x=23.3752005339$

$$\text{solve}\left(\frac{-419.40510603576\cdot\left(x^2-d\cdot(d+7.32)\right)}{x^4+2\cdot\left(d^2+7.32\cdot d+26.7912\right)\cdot x^2+d^2\cdot(d+7.32)^2}=0,x\right)\Big|d=10$$

$x=-13.1605471011$ or $x=13.1605471011$

$$\text{solve}\left(\frac{-419.40510603576\cdot\left(x^2-d\cdot(d+7.32)\right)}{x^4+2\cdot\left(d^2+7.32\cdot d+26.7912\right)\cdot x^2+d^2\cdot(d+7.32)^2}=0,x\right)\Big|d=5$$

$x=-7.84856674814$ or $x=7.84856674814$

$$\text{solve}\left(\frac{-419.40510603576\cdot\left(x^2-d\cdot(d+7.32)\right)}{x^4+2\cdot\left(d^2+7.32\cdot d+26.7912\right)\cdot x^2+d^2\cdot(d+7.32)^2}=0,x\right)\Big|d=15$$

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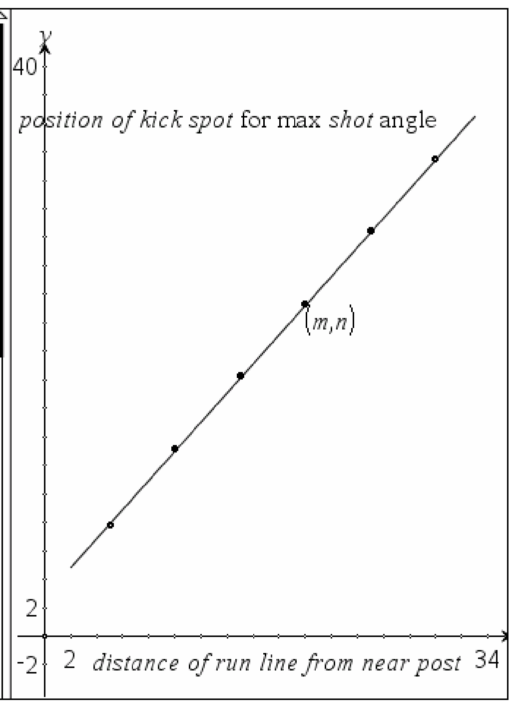
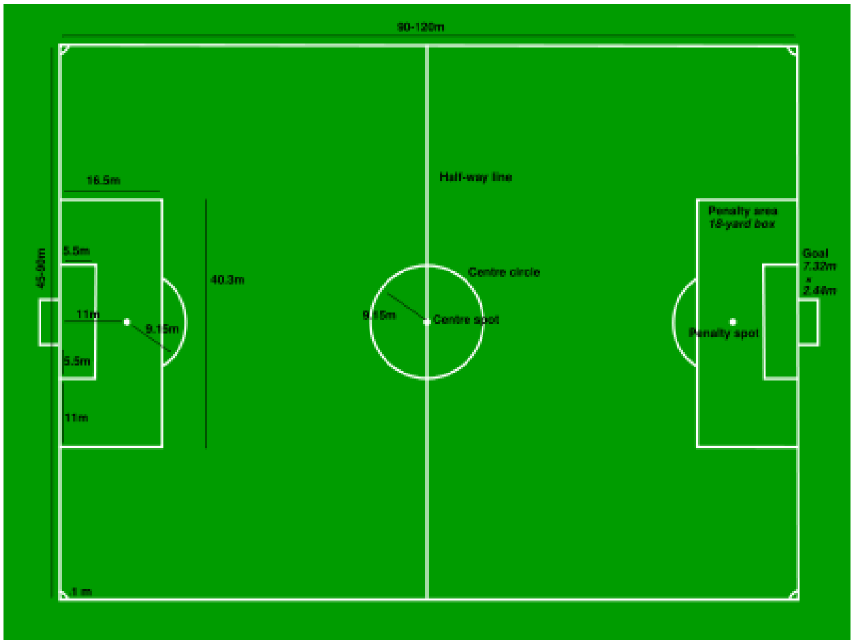
The position of the maximum angle kicking spot on different run lines, as measured from their distance from the near post of the goal.

	A m	B n	C
◆			
1	5	7.84856674814	
2	10	13.1605471011	
3	15	18.2975408184	
4	20	23.3752005339	
5	25	28.4253408071	
6	30	33.4604243846	
7			
8			
9			
10			
11			
12			
13			
14			
15			



The Dimensions of a FOOTBALL FIELD



Goal opening = 7.32 metres (8 yards)

Length = 90 to 120 metres

Width = 45 to 90 metres (60 yards)