SURDS, SPIRALS, DYNAMIC GEOMETRY AND CAS

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This paper aims to enrich the teaching of surds. We use explorations with spiral and other geometric patterns, supported by dynamic geometry, to link surds with length through Pythagoras' theorem, emphasise their relative size and to provide attractive contexts for simple and complex surd calculations. The exact value mode of CAS may support students' work with surds, allowing them to construct patterns and share a world of beautiful mathematics.

Learning about surds is an important milestone for students. With the possible exception of π, surds introduce students to a new class of numbers – the irrational numbers. According to legend, the followers of Pythagoras, who lived around 500BC on the Greek island of Samos, were the first people to realize that what people considered to be numbers at that time (i.e. the rational numbers, our natural numbers and common fractions) were not able to describe the lengths of all lines. Irrational numbers arise from simple geometric situations; if the two sides of a right angled isosceles triangle each have length one unit, then the hypotenuse has length $\sqrt{2}$ units. It is very likely that the observation that the rational numbers were not enough to describe lengths, was made in such as context like this using Pythagoras' theorem. This theorem had already been known for about a thousand years, but perhaps Pythagoras was the first to understand the importance of a proof, as he is regarded as the first pure mathematician (HREF1). Again according to legend, the finding that there was no number (remember, at that time, this meant rational number) to express the length of this line so disturbed the worldview of the Pythaogoreans whose guiding principle was"all is number", that they swore members of their group not to reveal this to outsiders. When the secret was betrayed, the perpetrator was drowned at sea. These historical links make it fitting that learning about surds should be strongly linked with geometry, as in the introductory explorations suggested here.

This article presents four explorations in geometry which provide a context for students to learn about surds and to practice manipulations. The explorations can be done using printed worksheets, with a ruler for measuring lengths and calculator for working out decimal approximations. However if geometry software is available, then students have access to automatic measuring tools, including for the measurement of area. The diagrams can also easily be produced in colour, which makes the explorations more attractive. In addition, if students have access to a CAS computer software or calculator (e.g. Texas Instruments TI89, the Casio FX&) or the Hewlett Packard HP40G), then they make good use of these to scaffold or check their exact calculations. These activities have been developed as part of the RITEMATHS project (HREF2), which is a research project of the University of Melbourne, six secondary schools and funded by the Australian Research Council. Worksheets and software files can be accessed from the project website.

Although surds have their historical origins in geometry, they are not intended for practical measurement. Surds belong to the world of theoretical mathematics, and not to the world of practical mathematics for everyday life and, in particular, not for measuring. By using the surds arising from the application of Pythagoras' theorem, we can calculate the length of a ladder against a wall, but the result in terms of surds is useless for a practical purpose until it is converted to a numerical approximation. When students learn about surds, they are making a transition from practical arithmetic to the world of mathematics. With the advent of computer algebra systems (CAS), students can also be supported in this theoretical world, just as numerical calculators have supported students in their work with integers and finite decimals.

Learning about surds requires some substantial conceptual adjustment and conceptual expansion, in addition to the manipulative skills that become the focus of most of the exercises that students do. Surds, although they have an exact value, have decimal expansions which cannot readily be described, unlike the terminating or repeating decimal expansions of rational numbers. Students need to learn that a surd has an exact and known value, despite the fact that the digits in its decimal expansion can never be calculated. Later, students will come across many other irrational numbers, but the surds are the first and prototypical class.

Explorations 1 and 2 – meaning, values and simple manipulation

Explorations 1 and 2 are both based around finding the lengths of lines in spirals constructed geometrically. Spiral One (see Figure 1) begins with an isosceles triangle with two sides of length one (i.e. side lengths, 1,1, $\sqrt{2}$). Another right angle triangle is built on this one, using the hypotenuse of the previous triangle as one side, and a segment of length one as the other (i.e. side lengths $1,\sqrt{2},\sqrt{3}$). A third triangle is constructed, again using the hypotenuse of the previous triangle and a perpendicular line of length one (side lengths $1,\sqrt{3},\sqrt{4}=2$). The spiral continues as shown in Figure 1, so that the outer side lengths are always one. This spiral can be given to students on paper, and also in a Geometer's Sktechpad file. The task for students is to find the length of each line segment in the spiral, as an exact value (integer or surd) and as a decimal. Students could measure the segments using the "measure" command in GSP, or using a ruler from the paper copy. They can also use the 'calculate command' in Geometer's Sketchpad to convert from sexact to approximate numbers.

The exact lengths of the hypotenuses of the triangles in Spiral One are $\sqrt{2},\sqrt{3},\sqrt{4}=2,\sqrt{5},\sqrt{6},\sqrt{7},\sqrt{8},\sqrt{9}=3$. These are easy to calculate because only simple square roots need to be squared. A typical calculation (e.g. for the fifth triangle is $hypotenuse=\sqrt{1^2+\sqrt{5}^2}=\sqrt{1+5}=\sqrt{6}$.)

This introductory exercise is valuable for students to learn that surds have a definite value, represented by the length of the line segment, and that they are exact, but can be approximated. When we trialled it, students liked to work in Geometer's Sketchpad, where the spiral was attractively coloured, and they could use the measurement tool. If CAS is available, then students can work in the exact mode to apply Pythagoras' theorem, using the CAS as "trainer wheels" to check their exact calculations.

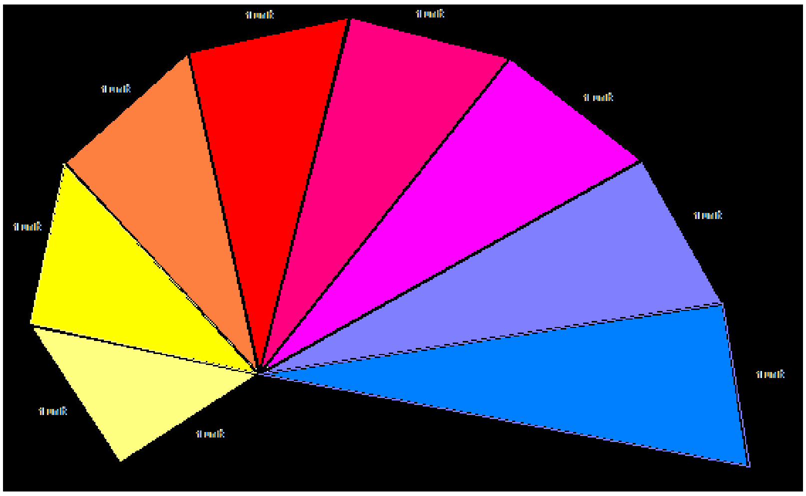


Figure 1 Spiral One – A sequence of right angled triangles with outer side always equal to one.

Spiral Two requires the same manipulative skills of squaring simple surds (such as $\sqrt{2}$) and taking square roots. However, it can also introduce taking square factors from surds (e.g. writing $\sqrt{8} = 2\sqrt{2}$). The spiral is made by constructing isosceles triangles at each stage (see Figure 2), beginning with an isosceles triangle with two sides of length one (i.e. sides 1,1, $\sqrt{2}$) . Another right angle isosceles triangle is built on this one, using the hypotenuse of the previous triangle as one side (i.e. sides of length $\sqrt{2}$, $\sqrt{2}$, 2). A third right angle isosceles triangle is constructed, again using the hypotenuse of the previous triangle as one side (i.e. side lengths 2, 2, $2\sqrt{2}$). The spiral continues as shown in Figure 2, so that the outer side lengths are always one. This spiral can be given to students on paper, and also in a Geometer's Sktechpad file. Again, the task for students was to find the length of each line segment in the spiral, as an exact value (integer or surd) and as a decimal. Students could measure the segments using the "measure" command in GSP, or using a ruler from the paper copy and they can also calculate the approximate answers on a calculator or using the "calculate" command in Geometer's Sketchpad. The sequence of lengths of the hypotenuses is $\sqrt{2}, \sqrt{4} = 2, \sqrt{8} = 2\sqrt{2}, \sqrt{16} = 4, \sqrt{32} = 2\sqrt{2}$ … In this spiral, the relationships between the various square roots (e.g. that $\sqrt{8}$ = $2\sqrt{2}$) can be seen in the numerical answers, e.g. that $\sqrt{8}$ = 2.828427125… and $\sqrt{2}$ = 1.414213562…

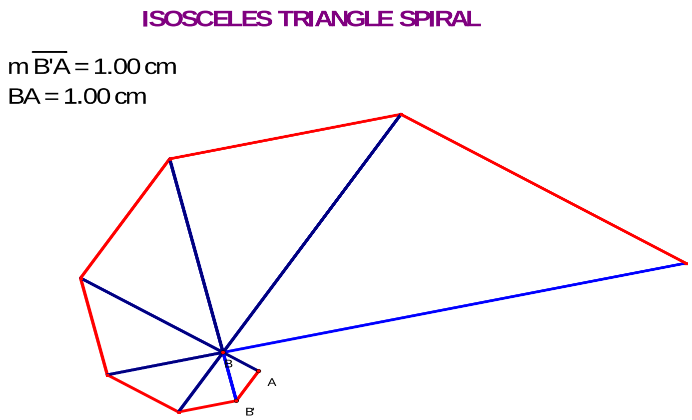


Figure 2 Spiral Two – A sequence of right angled isosceles triangles.

Exploration 3

The tessellation which forms the basis of the design at Federation Square in the centre of Melbourne provides an opportunity to use slightly more advanced skills. The photo in Figure 3 shows that the basic tiles in Federation Square are right angle triangles all of the same shape (actually having sides in the ratio of 1 : 2 : $\sqrt{5}$). Five of these combine to make a larger triangle, which is geometrically similar. Because of this fact, in turn five of the larger triangles can combine to form an even larger triangle, which is again geometrically similar. Vincent (2003) explains more about the architecture and more about the properties of this 'pinwheel' tiling pattern. The worksheet in Figure 4 can be presented on paper, or as a Geometer's Sketchpad file. Students are asked to find the lengths of the sides of the various triangles. Again students can proceed by measuring using the Sketchpad tool, by measuring with a ruler on a paper copy, and doing the calculations by hand or using CAS to scaffold and check their exact calculations.

The skills involved in these calculations are a little more advanced than in Explorations 1 and 2, and since there are many geometric relationships, there are possibilities for finding and using various properties of surds. For example, the lengths of the perpendicular sides of the second triangle in Figure 4 are $\sqrt{5}$ (the shortest side of second triangle, calculated as the hypotenuse of the smallest triangle), and $\sqrt{5}+\sqrt{5}$. Finding the length of the hypotenuse requires the following calculations:

$$(\sqrt{5}+\sqrt{5}) = 2\sqrt{5}$$

$$(2\sqrt{5})^2 = 2\sqrt{5}\times 2\sqrt{5} = 2\times 2\times \sqrt{5}\times\sqrt{5} = 4\times 5 = 20$$

$$(\sqrt{5})^2 + (2\sqrt{5})^2 = 5+20 = 25$$

$$hypotenuse = \sqrt{25} = 5$$

The fact that the hypotenuse of the second triangle has length 5 can also be seen directly from the fact that it is made of three segments of length 1, 2 and 2. In this exploration, the challenge is to find the lengths in many ways, and to see that they give the same answers. Students may also be able to observe that the fact that the smallest and the second triangle triangles are definitely geometrically similar because their sides are in the same ratio (1 : 2: $\sqrt{5}$ = $\sqrt{5}$: 2$\sqrt{5}$: 5). Federation Square provides many opportunities for geometry with a local flavour – this is just one possibility. Investigations of similar triangles are another good choice.

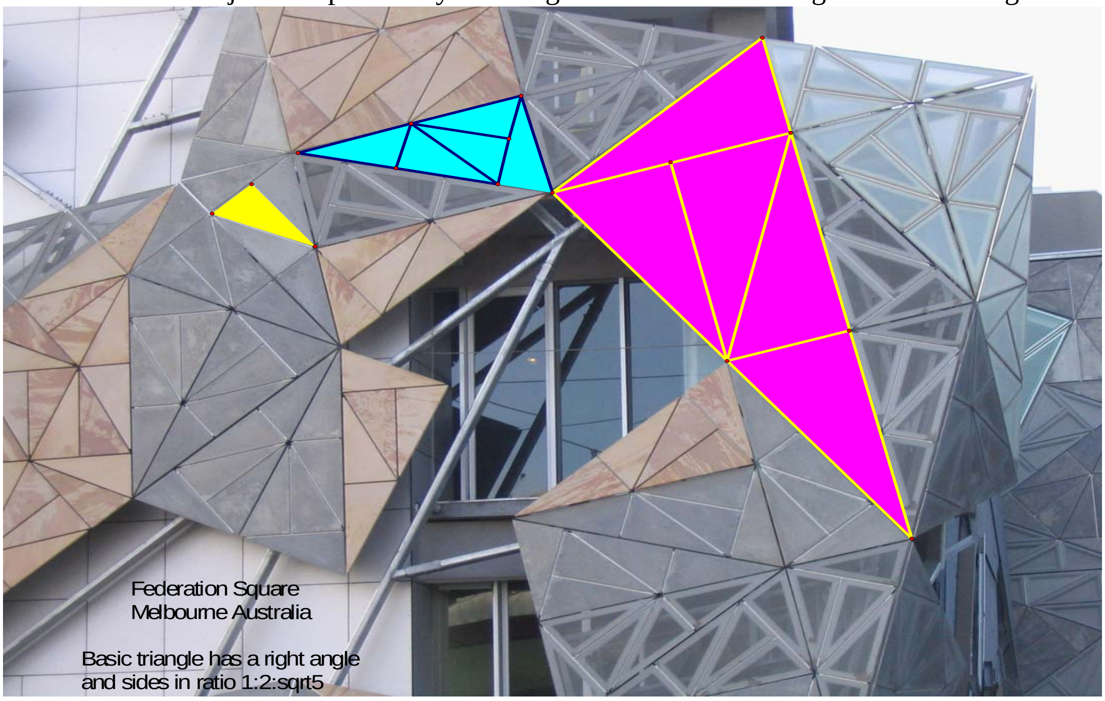


Figure 3 Federation Square, Melbourne, showing how the pattern is constructed from tessellating triangles.

Exploration 4

The square root of 5 is also crucial to the next exploration, which is based on the golden ratio

$$\varphi = \frac{1}{2}(\sqrt{5}+1)$$

The golden ratio (φ, pronounced 'phi') is said to be the ratio of width to height of the rectangle of most pleasing shape. It occurs in nature, often through its intimate links with the Fibonacci sequence. Ron Knott's website on Fibonacci Numbers and the Golden Section (HREF3) contains many multimedia demonstrations of the occurrence of the golden ratio in nature. Exploration 4 is based on the fact that if a square is cut off a golden rectangle (i.e. one with sides in the golden ratio), then the remaining piece is also a golden rectangle. The exploration asks students to find the lengths, areas, and ratio of sides of a sequence of rectangles. Because the first rectangle is a golden rectangle, we can prove in this activity that all the subsequent rectangles are golden as well. Each of these investigations can be done numerically at any level, but exact arithmetic will need appropriate preparation.

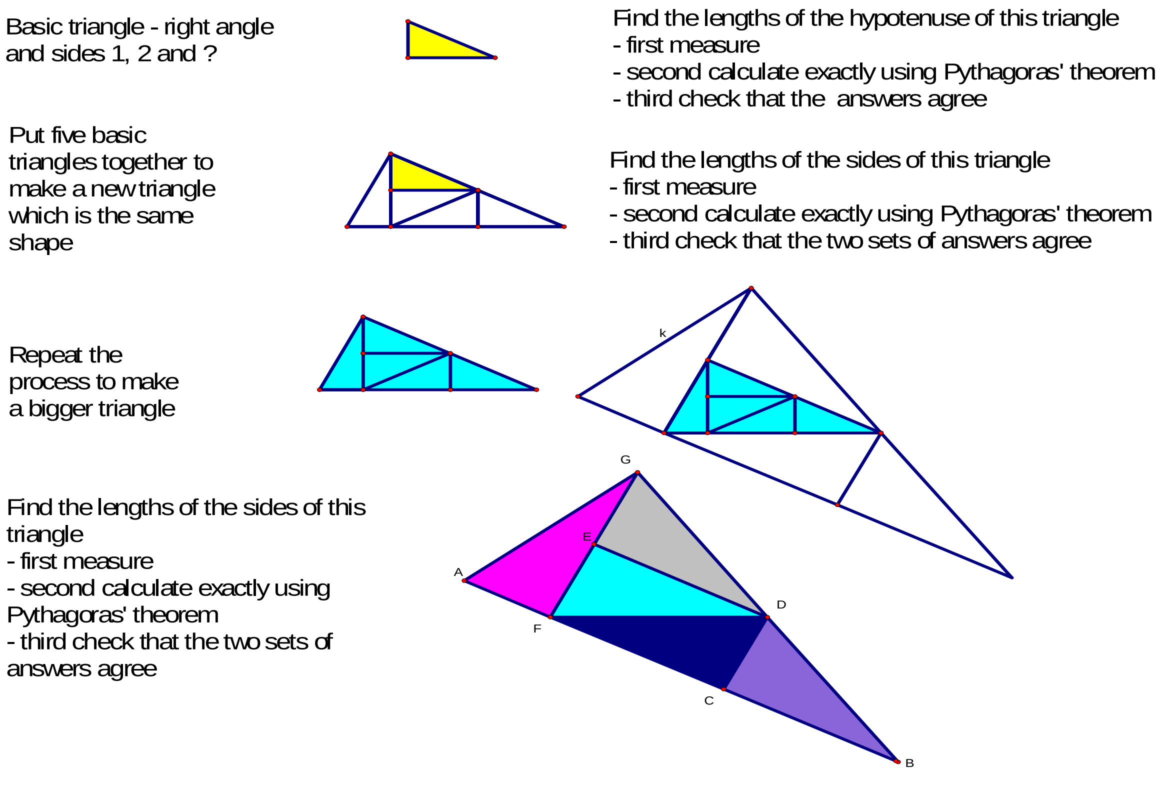


Figure 4 Worksheet on Federation Square triangles, which is presented in Geometer's Sketchpad.

Figure 5 shows a golden rectangle ABCD. The points E and F are constructed so that AB = BF= FE = EA to make a square, which is then removed. The remaining part of ABCD is the rectangle CDEF and it can be shown that this is also a golden rectangle. This is the hardest calculation in Exploration 4 – see below. The process of removing squares from the golden rectangles can continue indefinitely, constructing in turn the progressively smaller golden rectangles CIJF, FGHJ and JKLH and so on.

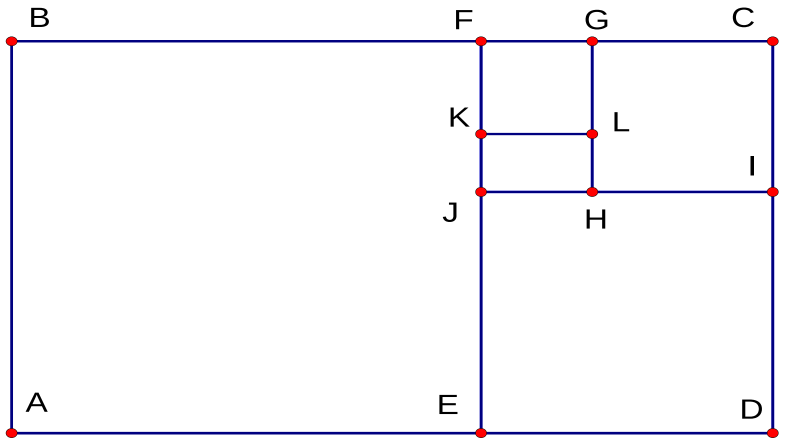


Figure 5 A golden rectangle, ABCD, dissected into progressively smaller golden rectangles (CDEF, CIJF, FGHJ and JKLH).

Exploration 4 starts with a golden rectangle with sides measuring $2\sqrt{5}$ (length AB) and $5+\sqrt{5}$ (length BC). These dimensions have been selected to make a golden rectangle that is a good size to print and measure. Finding the side lengths of the next rectangle (CDEF) requires calculation with composite expressions involving surds:

$$(5+\sqrt{5})-2\sqrt{5}=5-\sqrt{5}$$

and the side length of the next rectangle requires dealing with a double minus:

$$2\sqrt{5}-(5-\sqrt{5})=3\sqrt{5}-5.$$

Again, these calculations can be carried out in parallel with measurements from the diagrams, and scaffolded where required by using CAS in exact mode and converting exact answers to decimal approximations on the calculator.

Finding the areas of the successive golden rectangles provides an opportunity to learn to multiply complex expressions, or practice previously learned skills. For example, the area at step 3 is equal to

$$(5-\sqrt{5})\times(3\sqrt{5}-5)=15\sqrt{5}-25-3\times5+5\sqrt{5}=20\sqrt{5}-40$$

By creating the quadrilateral interior in Geometer's Sketchpad, the area can also be measured to be equal to 4.72 cm^2. CAS can be used in exact mode to assist with the surd calculation and a decimal approximation to the answer can be compared with the measurement. Figure 7 shows the worksheet.

Showing that all of the rectangles are geometrically similar, i.e. that the ratio of the sides is always equal to φ, requires advanced school level skills. Students can easily find that the ratios are numerically approximately equal to 1.68 , but to show that the rectangles are all golden rectangles, the following ratios need to be shown to be equal:

$$\frac{\sqrt{5}+5}{2\sqrt{5}}=\frac{2\sqrt{5}}{5-\sqrt{5}}=\frac{5-\sqrt{5}}{3\sqrt{5}-5}=\frac{3\sqrt{5}-5}{10-4\sqrt{5}}$$

It is here that the skill of rationalizing a denominator is very useful, because it allows these unwieldy ratios which certainly do not LOOK equal to be readily compared, as illustrated for three cases in Figure 6. In the distant past, when numerical calculation was difficult, rationalizing the denominator was an important skill for practical arithmetic. Who would wish to divide by 5-$\sqrt{5}$ (i.e. 2.76393202250021etc) when you can divide instead by 2 as shown in the second line of Figure 6? This example shows how rationalizing the denominator remains important in theoretical mathematics, even now that its use in practical calculation has been eclipsed.

$$\frac{\sqrt{5}+5}{2\sqrt{5}}=\frac{\sqrt{5}+1}{2}$$

$$\frac{2\sqrt{5}}{5-\sqrt{5}}=\frac{2\sqrt{5}\times(5+\sqrt{5})}{(5-\sqrt{5})\times(5+\sqrt{5})}=\frac{10\sqrt{5}+10}{25-5}=\frac{10\sqrt{5}+10}{20}=\frac{\sqrt{5}+1}{2}$$

$$\frac{5-\sqrt{5}}{3\sqrt{5}-5}=\frac{5-\sqrt{5}}{3\sqrt{5}-5}\times\frac{3\sqrt{5}+5}{3\sqrt{5}+5}=\frac{15\sqrt{5}+25-15-5\sqrt{5}}{45-25}=\frac{10\sqrt{5}+10}{20}=\frac{\sqrt{5}+1}{2}$$

Figure 6 Rationalising the denominators shows that all of the rectangles are golden.

Using CAS can support most of the work with division of expressions involving surds. For example, the TI Voyage 200 produces from the entry $\frac{2\sqrt{5}}{5-\sqrt{5}}$ (line 2, Figure 6) the response $\frac{\sqrt{5}}{2}+\frac{1}{2}$, which is obtained by rationalizing the denominator as shown above. However, beginners will also find it difficult to control the form of expressions easily. The entry of $\frac{1}{\sqrt{2}}$, for example, immediately produces $\frac{\sqrt{2}}{2}$, but it is very hard to go the other way if this is not the desired form.

Conclusion

The teaching of surds provides many opportunities for investigations which help students make the concept-based and skill-based transitions from practical arithmetic to theoretical mathematics. The curriculum links with Pythagoras' theorem are very strong, and so using geometry to teach about surds is natural. The explorations have not covered all aspects of surds. The ability of CAS to make prime factorizations of numbers, for example, might be used to scaffold learning about properties that have not been properly covered here, such as $\sqrt{ab}=\sqrt{a}.\sqrt{b}$ and how to find the square root of a number from its prime factorization. The explorations above how shown how technology can enhance the teaching and learning. The geometry software adds colour, aids in the construction of diagrams, and makes measuring easy, including for areas, but it is not necessary. The CAS capability has been used mainly to scaffold student's learning of manipulative skills, so that they build confidence and can check their work. The easy links to the numerical values of exact expressions provide a strong reminder for students that surds are numbers with definite values, even though their decimal expansions are not known.

References:

HREF1 http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Pythagoras.html Pythagoras of Samos. Accessed 20 June 2005

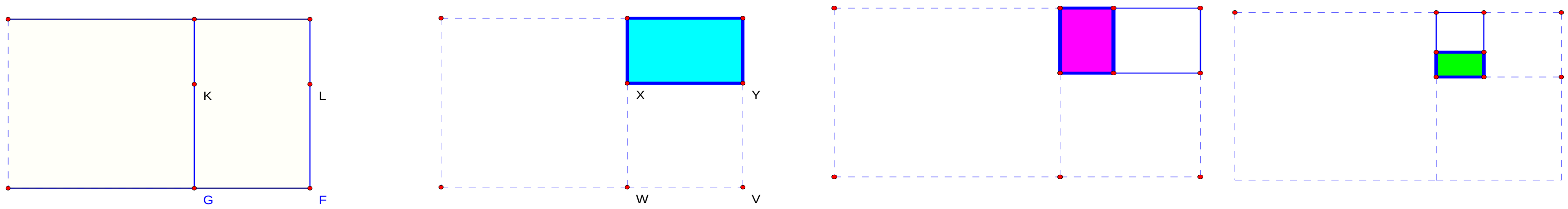
HREF2 http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS Accessed 2 June 2005

HREF3 http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html Fibonacci Numbers and the Golden Section.Accessed 20 June 2005

Vincent, Jill (2003) Shrine to University: A geometry journey along St Kilda Road and Swanston Street (2nd edition). Melbourne: Mathematical Association of Victoria.

Figure 7. Golden Rectangle Investigation

(Take measurements from the GSP file *GoldenRectangleRITEMATHS*, not from this page)



Rect-angle No.	Exact Calculations				Measurements (approximate)			
	Longer side	**Shorter side**	**Area**	**Longer: shorter ratio**	**Longer side**	**Shorter side**	**Area**	**Longer: shorter ratio**
1 (first large size)	$5+\sqrt{5}$	$2\sqrt{5}$						
2 (yellow)								
3 (blue)								
4 (purple)								
5 (green)								
6 (keep going)								
7								