Parabola Problem
(CAS)

(a) Points O(0, 0) and P(m,n) are two fixed points with m > 0 and n > 0. There is a family of parabolas that pass through *O* and *P* with their maximum to the left of *P*. Each parabola in this family will intersect the x-axis in a point Q(k,0) for some *k*. Find a general rule for the family of parabolas. What restrictions must be placed on the number values for *k*?

$$\text{Define } f(x) = a\cdot x^2 + b\cdot x + c$$

"Done"

$$\text{Solve}(f(0) = 0 \text{ and } f(m) = n \text{ and } f(k) = 0, \{a, b, c\}) \mid m > 0 \text{ and } n > 0 \text{ and } k > 0$$

$$a = \frac{-n}{(k-m)\cdot m} \text{ and } b = \frac{k\cdot n}{(k-m)\cdot m} \text{ and } c = 0$$

$$\text{Define } f(x) = a\cdot x^2 + b\cdot x + c \mid ans$$

"Done"

$$f(x)$$

$$\frac{k\cdot n\cdot x}{(k-m)\cdot m} - \frac{n\cdot x^2}{(k-m)\cdot m}$$

The coordinates of the maximum point of the parabola are found as below:

$$\text{fmax}(f(x), x) \mid m > 0 \text{ and } n > 0 \text{ and } k > m$$

$$x = \infty \text{ or } x = -\infty \text{ or } x = \frac{k}{2}$$

$$f(x) \mid x = \frac{k}{2}$$

$$\frac{k^2\cdot n}{4\cdot(k-m)\cdot m}$$

The maximum value must be more than n.

$$\text{solve}(ans > n, k) \mid n > 0$$

$$\frac{m\cdot n}{k-m} + \frac{k\cdot n}{m} - 3\cdot n > 0$$

$$\text{comDenom}(ans)$$

$$\frac{k^2\cdot n - 4\cdot k\cdot m\cdot n + 4\cdot m^2\cdot n}{k\cdot m - m^2} > 0$$

$$\text{factor}(ans, k)$$

$$\frac{(k-2\cdot m)^2\cdot n}{(k-m)\cdot m} > 0$$

But n > 0 and m >0 so,

$\frac{ans}{n} \mid n > 0$

$\frac{(k-2\cdot m)^2}{(k-m)\cdot m} > 0$

$ans\cdot m \mid m > 0$

$\frac{(k-2\cdot m)^2}{k-m} > 0$

The numerator is always positive, so for the result above to be positive we must have k - m >0, so we must have k > m.

And for the result above to not equal zero we must have k < 2m or k > 2m which means k/2 < m or k/2 >m. If k/2 > m, the maximum of f(x), which occurs when x = k/2, occurs at value more than m, so the maximum would lie to the right of the point (m,n). Therefore, k < 2m.

(b) Find the general rule for the family of parabolas that are the reflection in the line x = m of the parabolas in part (a).

The reflected parabola will go through the reflected images of (0,0) , (k,0) and (m, n) in the vertical line x = m. That is through the points (2m,0) , (2m - k,0) and (m, n). The general rule for the family of parabolas through these points is found as follows:

Define $h(x) = a\cdot x^2 + b\cdot x + c$

"Done"

$\text{Solve}(h(2\cdot m) = 0 \text{ and } h(2\cdot m - k) = 0 \text{ and } h(m) = n, \{a, b, c\}) \mid k < 2\cdot m \text{ and } k > 0$

$a = \frac{-n}{(k-m)\cdot m} \text{ and } b = \frac{-(k-4\cdot m)\cdot n}{(k-m)\cdot m} \text{ and } c = \frac{2\cdot(k-2\cdot m)\cdot n}{k-m}$

Define $h(x) = a\cdot x^2 + b\cdot x + c \mid ans$

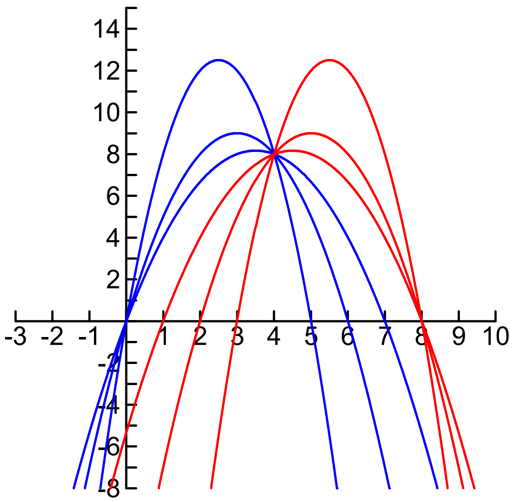
"Done"

$h(x)$

$\frac{-n\cdot x^2}{(k-m)\cdot m} - \frac{(k-4\cdot m)\cdot n\cdot x}{(k-m)\cdot m} + \frac{2\cdot(k-2\cdot m)\cdot n}{k-m}$

(c) For m = 4 and n = 8 choose any three of the possible values for *k*. With the aid of a graphing or a CAS calculator draw on the same set of axes the graph of the parabola and its reflection for each of the values of *k*.

We choose k = 5, 6, 7 as the values. Then the graphs for y = f(x) and y = h(x) for these values with m = 4 and n = 8 are found below.



(d) The maximum points for all the parabolas in part (a) lie on the graph of a function. Find the rule for this function.

The x-coordinate of the maximum point is given by x = k/2, so k = 2x and the y-coordinate by:

$$f(x) \mid x = \frac{k}{2}$$

$$\frac{k^2 \cdot n}{4 \cdot (k - m) \cdot m}$$

Then the rule for the required function is given by,

$$\text{Define } g(x) = ans \mid k = 2 \cdot x$$

"Done"

$$g(x)$$

$$\frac{n \cdot x^2}{m \cdot (2 \cdot x - m)}$$