Parabola Problem  
(CAS)

(a) Points \(O(0, 0)\) and \(P(m,n)\) are two fixed points with \(m > 0\) and \(n > 0\). There is a family of parabolas that pass through \(O\) and \(P\) with their maximum to the left of \(P\). Each parabola in this family will intersect the x-axis in a point \(Q(k,0)\) for some \(k\).

Find a general rule for the family of parabolas. What restrictions must be placed on the number values for \(k\)?

Define \(f(x) = ax^2 + bx + c\)

"Done"

Solve \((f(0) = 0 \quad \text{and} \quad f(m) = n \quad \text{and} \quad f(k) = 0, \{a, b, c\})\) \(| m > 0\) and \(n > 0\) and \(k > 0\)

\[
a = \frac{-n}{(k - m)m} \quad \text{and} \quad b = \frac{kn}{(k - m)m} \quad \text{and} \quad c = 0
\]

Define \(f(x) = ax^2 + bx + c\) | ans

"Done"

\[
f(x) = \frac{k \cdot n \cdot x}{(k - m)m} - \frac{n \cdot x^2}{(k - m)m}
\]

The coordinates of the maximum point of the parabola are found as below:

\[
\text{max}(f(x), x) \quad | \quad m > 0 \quad \text{and} \quad n > 0 \quad \text{and} \quad k > m
\]

\[
x = \infty \quad \text{or} \quad x = -\infty \quad \text{or} \quad x = \frac{k}{2}
\]

\[
f(x) \quad | \quad x = \frac{k}{2}
\]

The maximum value must be more than \(n\).

solve \((\text{ans} > n, k) \quad | \quad n > 0\)

\[
\text{comDenom(ans)}
\]

\[
\frac{m \cdot n}{k - m} + \frac{k \cdot n}{m} - 3 \cdot n > 0
\]

\[
\frac{k^2 \cdot n - 4 \cdot k \cdot m \cdot n + 4 \cdot m^2 \cdot n}{k \cdot m - m^2} > 0
\]

factor \((\text{ans}, k)\)

\[
\frac{(k - 2 \cdot m)^2 \cdot n}{(k - m)m} > 0
\]
But $n > 0$ and $m > 0$ so,

$$\frac{n}{n} \cdot \frac{(k - 2m)^2}{(k - m)m} > 0 \quad \text{and} \quad \frac{(k - 2m)^2}{k - m} > 0$$

The numerator is always positive, so for the result above to be positive we must have $k - m > 0$, so we must have $k > m$.

And for the result above to not equal zero we must have $k < 2m$ or $k > 2m$ which means $k/2 < m$ or $k/2 > m$. If $k/2 > m$, the maximum of $f(x)$, which occurs when $x = k/2$, occurs at value more than $m$, so the maximum would lie to the right of the point $(m, n)$. Therefore, $k < 2m$.

(b) Find the general rule for the family of parabolas that are the reflection in the line $x = m$ of the parabolas in part (a).

The reflected parabola will go through the reflected images of $(0,0)$, $(k,0)$ and $(m, n)$ in the vertical line $x = m$. That is through the points $(2m,0)$, $(2m - k,0)$ and $(m, n)$. The general rule for the family of parabolas through these points is found as follows:

Define $h(x) = a x^2 + b x + c$

"Done"

Solve $h(2m) = 0$ and $h(2m - k) = 0$ and $h(m) = n$, $\{a, b, c\}$ | $k < 2m$ and $k > 0$

$$a = \frac{-n}{(k - m)m} \quad \text{and} \quad b = \frac{-(k - 4m)n}{(k - m)m} \quad \text{and} \quad c = \frac{2(k - 2m)n}{k - m}$$

Define $h(x) = a x^2 + b x + c$ | ans

"Done"

$$h(x) = \frac{-n x^2}{(k - m)m} - \frac{(k - 4m)n x}{(k - m)m} + \frac{2(k - 2m)n}{k - m}$$
(c) For \( m = 4 \) and \( n = 8 \) choose any three of the possible values for \( k \). With the aid of a graphing or a CAS calculator draw on the same set of axes the graph of the parabola and its reflection for each of the values of \( k \).

We choose \( k = 5, 6, 7 \) as the values. Then the graphs for \( y = f(x) \) and \( y = h(x) \) for these values with \( m = 4 \) and \( n = 8 \) are found below.

(d) The maximum points for all the parabolas in part (a) lie on the graph of a function. Find the rule for this function.

The x-coordinate of the maximum point is given by \( x = k/2 \), so \( k = 2x \) and the y-coordinate by:

\[
f(x) \big| x = \frac{k}{2}
\]

\[
\frac{k^2 - n}{4(k - m) \cdot m}
\]

Then the rule for the required function is given by,

\[
g(x)
\]

Define \( g(x) = \text{ans} \mid k = 2x \)

"Done"

\[
\frac{n \cdot x^2}{m \cdot (2x - m)}
\]